REAL COHOMOLOGY AND THE POWERS OF THE FUNDAMENTAL IDEAL IN THE WITT RING

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Abstract. Let A be a local ring with 2 invertible. It is known that the localization of the cohomology ring $H^*_{ét}(A, \mathbb{Z}/2)$ with respect to the class $(-1) \in H^1_{ét}(A, \mathbb{Z}/2)$ is isomorphic to the ring $\mathrm{C}(\operatorname{sper} A, \mathbb{Z}/2)$ of continuous $\mathbb{Z}/2$-valued functions on the real spectrum of A. Let $I^n(A)$ denote the powers of the fundamental ideal in the Witt ring of symmetric bilinear forms over A. The starting point of this article is the "integral" version: the localization of the graded ring $\bigoplus_{n\geq 0} I^n(A)$ with respect to the class $\langle\langle -1\rangle\rangle := \langle 1, 1\rangle \in I(A)$ is isomorphic to the ring $\mathrm{C}(\operatorname{sper} A, \mathbb{Z})$ of continuous $\mathbb{Z}$-valued functions on the real spectrum of A.

This has interesting applications to schemes. For instance, for any algebraic variety X over the field of real numbers $\mathbb{R}$ and any integer n strictly greater than the Krull dimension of X, we obtain a bijection between the Zariski cohomology groups $H^*_{Zar}(X, \mathcal{I}^n)$ with coefficients in the sheaf $\mathcal{I}^n$ associated to the nth power of the fundamental ideal in the Witt ring $W(X)$ and the singular cohomology groups $H^*_{sing}(X(\mathbb{R}), \mathbb{Z})$.

1. Introduction

Let X be an algebraic variety over the field of real numbers and let d denote the Krull dimension of X. Let $\mathcal{H}^n$ denote the Zariski sheaf associated to the presheaf $U \mapsto H^n_{ét}(U, \mathbb{Z}/2)$, where $H^n_{ét}(U, \mathbb{Z}/2)$ denotes the étale cohomology of U with $\mathbb{Z}/2\mathbb{Z}$-coefficients. Under the hypotheses that X is smooth, integral, and quasi-projective a classic theorem of Raman Parimala and Jean-Louis Colliot-Thélène states that the sections of $\mathcal{H}^n$ are in bijection with $H^0_{sing}(X(\mathbb{R}), \mathbb{Z}/2)$ when $n \geq d+1$ [CTP90, Theorem 2.3.1]; it follows from this that there is a bijection of cohomology groups

$$H^*_{Zar}(X, \mathcal{H}^n) \simeq H^*_{sing}(X(\mathbb{R}), \mathbb{Z}/2) \tag{1}$$

when $n \geq d+1$, where $X(\mathbb{R})$ denotes the real points of X equipped with the Euclidean topology (Remark 4.3 defines this topology) and $H^*_{sing}(X(\mathbb{R}), \mathbb{Z}/2)$ denotes the singular cohomology groups of the real points with $\mathbb{Z}/2\mathbb{Z}$-coefficients.

Let $W(X)$ denote the Witt ring of symmetric bilinear forms over X and $I^n(X)$ the powers of the fundamental ideal c.f. [Kne77]. Let $\mathcal{I}^n$ denote the Zariski sheaf associated to the presheaf $U \mapsto I^n(U)$. Let $\overline{\mathcal{I}^n}$ denote the sheaf associated to the

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presheaf $U \mapsto \mathcal{I}^n(U)/\mathcal{I}^{n+1}(U)$. The short exact sequence of sheaves

$$0 \to \mathcal{I}^{n+1} \to \mathcal{I}^n \to \overline{\mathcal{I}^n} \to 0$$

induces a long exact sequence in Zariski cohomology

(2)

$$\cdots \to H^m_{Zar}(X, \mathcal{I}^{n+1}) \to H^m_{Zar}(X, \mathcal{I}^n) \to H^m_{Zar}(X, \overline{\mathcal{I}^n}) \xrightarrow{\partial} H^{m+1}_{Zar}(X, \mathcal{I}^{n+1}) \to \cdots$$

In the introduction to [Fas13] Jean Fasel made the following assertions: the Zariski cohomology groups $H^*_{Zar}(X, \mathcal{I}^n)$ are the analogue of the singular cohomology groups $H^*_{sing}(X(\mathbb{R}), \mathbb{Z})$, while $H^*_{Zar}(X, \overline{\mathcal{I}^n})$ are the analogue of $H^*_{sing}(X(\mathbb{R}), \mathbb{Z}/2)$; the map $H^*_{Zar}(X, \mathcal{I}^{n+1}) \to H^*_{Zar}(X, \mathcal{I}^n)$ corresponds to the homomorphism $H^*_{sing}(X(\mathbb{R}), \mathbb{Z}) \xrightarrow{2} H^*_{sing}(X(\mathbb{R}), \mathbb{Z})$ induced by the multiplication by 2 on the coefficients; the connecting homomorphism $H^*_{Zar}(X, \overline{\mathcal{I}}^n) \xrightarrow{\partial} H^*_{Zar}(X, \mathcal{I}^{n+1})$ is analogous to the Bockstein homomorphism $H^*_{sing}(X(\mathbb{R}), \mathbb{Z}/2) \xrightarrow{\beta} H^{*+1}_{sing}(X(\mathbb{R}), \mathbb{Z})$. Under the additional hypothesis that X is affine, smooth, and has trivial canonical sheaf, he proved that $H^d_{Zar}(X, \mathcal{I}^n) \simeq H^d_{sing}(X(\mathbb{R}), \mathbb{Z})$ for all $n \geq d$ [Fas11, Proposition 5.1].

We prove these assertions as a consequence of our more general results on real cohomology and the powers of the fundamental ideal. Precisely, we show in Corollary (8.3) that when $n \geq d+1$, the global signature induces an isomorphism $H^m_{Zar}(X, \mathcal{I}^n) \overset{\text{sign}}{\simeq} H^m_{sing}(X(\mathbb{R}), \mathbb{Z})$ for all $m \geq 0$ which in turn induces an isomorphism of long exact sequences from (2) to

$$\cdots \to H^m_{sing}(X(\mathbb{R}), \mathbb{Z}) \xrightarrow{2} H^m_{sing}(X(\mathbb{R}), \mathbb{Z}) \to H^m_{sing}(X(\mathbb{R}), \mathbb{Z}/2) \xrightarrow{\beta} H^{m+1}_{sing}(X(\mathbb{R}), \mathbb{Z}) \to \cdots$$

Real cohomology is a cohomology theory for schemes that globalizes to any scheme X singular cohomology in the sense that when X is a real variety, the real cohomology groups $H^m(X_r, \mathbb{Z})$ may be identified with the singular cohomology groups $H^m_{sing}(X(\mathbb{R}), \mathbb{Z})$. For details see Remark (4.3). The foundations and fundamental results on real cohomology are due to Claus Scheiderer [Sch94]. There is a close relationship between real and étale cohomology: the étale cohomology of X with 2-primary coefficients stabilizes in high degrees against the real cohomology of X with 2-primary coefficients [Sch94, Corollary (7.19), Proposition (19.8)]. He also obtained a generalization to schemes of the bijection (1). To introduce it, first recall that for any scheme X, multiplication by cup product with $(-1) \in H^1(X_{\acute{e}t}, \mathbb{Z}/2)$ induces a morphism of sheaves $\mathcal{H}^n \to \mathcal{H}^{n+1}$. Consequently one may consider the colimit $\varinjlim \mathcal{H}^n$ over the system

$$\mathcal{H}^0 \xrightarrow{(-1)} \mathcal{H}^1 \xrightarrow{(-1)} \mathcal{H}^2 \xrightarrow{(-1)} \cdots$$

The signature modulo 2 induces an isomorphism of sheaves $\varinjlim \mathcal{H}^n \to \operatorname{supp}_* \mathbb{Z}/2$ which induces an isomorphism of cohomology groups

$$H^m_{Zar}(X, \varinjlim \mathcal{H}^n) \simeq H^m(X_r, \mathbb{Z}/2) \tag{3}$$

for all $m \geq 0$, where $H^m(X_r, \mathbb{Z}/2)$ denotes the real cohomology of X with coefficients in the constant sheaf $\mathbb{Z}/2$ [Sch94, Corollary 19.5.1].

Note that one cannot obtain integral coefficient versions of the isomorphisms (1) and (3) by simply replacing everywhere $\mathbb{Z}/2$ with $\mathbb{Z}$ because when $n > d$ the étale cohomology groups $H^n_{\acute{e}t}(U, \mathbb{Z})$ are always torsion for any open subscheme U of X [Sch94, Corollary 7.23.3].

Here, we obtain integral versions by demonstrating in Theorem (8.5) that for any scheme X with two invertible in its global sections the signature induces an isomorphism of sheaves $\varinjlim \mathcal{I}^n \to \mathrm{supp}_* \mathbb{Z}$ which induces an isomorphism of cohomology groups

$$H^m_{Zar}(X, \varinjlim \mathcal{I}^n) \overset{\mathrm{sign}}{\simeq} H^m(X_r, \mathbb{Z})$$

for all $m \geq 0$, where $\varinjlim \mathcal{I}^n$ denotes the Zariski sheaf on X obtained by taking the colimit of the system of sheaves

$$\mathcal{W} \overset{\langle\langle -1\rangle\rangle}{\to} \mathcal{I} \overset{\langle\langle -1\rangle\rangle}{\to} \mathcal{I}^2 \overset{\langle\langle -1\rangle\rangle}{\to} \cdots$$

and $\mathcal{I}^n \overset{\langle\langle -1\rangle\rangle}{\to} \mathcal{I}^{n+1}$ denotes the map induced by tensor product with the Pfister form $\langle\langle -1\rangle\rangle := \langle 1, 1\rangle$.

These global results follow from the local case, that is, the statement on the localization of the graded ring $I^*(A)$ from the abstract. Another way of stating this is to say that

$$\text{(4)} \qquad \mathrm{sign} : \varinjlim I^n(A) \to \mathrm{C}(\mathrm{sper}\, A, \mathbb{Z})$$

is bijective for any local ring A with 2 invertible. Injectivity of (4) is well-known and follows from the local ring version of Pfister's local-global principal (for instance [Kne77, Ch. II §5] or directly in terms of the signature used in this article [Mah82, Théorème 2.1 and Corollaire]). The statement that (4) is surjective is stronger than Mahé's theorem, which states that the cokernel of sign $: W(A) \to \mathrm{C}(\mathrm{sper}\, A, \mathbb{Z})$ is 2-primary torsion for any commutative ring with 2 invertible. We believe it is known as well, but we don't know of a reference in the literature for surjectivity of (4) when A is local. We give a proof of bijectivity of (4) in Proposition (7.1) in a much different way using cohomological methods. For instance, in Theorem (5.2) we prove the Gersten conjecture for the Witt groups with 2 inverted of any regular excellent local ring. From this we deduce injectivity of (4) for any local ring with 2 invertible using "Hoobler's trick". Similarly, in Proposition (6.1) we prove a purity result for $\varinjlim I^n(A)$ in "geometric" cases and deduce surjectivity in general from this.

2. Total signature

For now and throughout this section, let F be a field of characteristic different from two, although the hypothesis on the characteristic is not necessary for the definitions.

Definition 2.1. An *ordering* on F is a subset $P \subset F$ satisfying the following:

(1) $P + P \subset P$, $PP \subset P$;
(2) $P \cap (-P) = 0$;
(3) $P \cup -P = F$.

If $b - a \in P$, then one writes $a \leq_P b$. If $a \in P$ and $a \neq 0$, then $a >_P 0$. It follows from the axioms that if F is nontrivial, then $1 >_P 0$. Also, for any $a \neq 0$ one writes $\mathrm{sgn}_P(a) = 1$ if $a \in P$ and $\mathrm{sgn}_P(a) = -1$ if $a \in -P$. From the axioms one has that $\mathrm{sgn}_P(ab) = \mathrm{sgn}_P(a)\mathrm{sgn}_P(b)$ for any $a, b \in F^\times$, consequently assigning any $a \in F^\times$ to $\mathrm{sgn}_P(a)$ determines a homomorphism $\mathrm{sgn}_P : F^\times \to \{\pm 1\}$ of groups. The pair (F, P) is called an *ordered field* [KS89, Kapitel I, Definition 1 and Bemerkungen].

Definition 2.2. The *real spectrum of* F, denoted $\operatorname{sper} F$, is the topological space formed by equipping the set of all orderings on F with the topology generated by the subbasis consisting of subsets $H(a) \subset \operatorname{sper} F$, $a \in F$, where $H(a)$ denotes the set of all orderings P satisfying $a >_P 0$.

Definition 2.3. Let P be an ordering on F. Any non-degenerate quadratic form ϕ over F splits as an orthogonal sum $\phi \simeq \phi_+ \perp \phi_-$, where the form ϕ_+ is positive definite with respect to the ordering (for all $0 \neq v$, $q(v) > 0$ with respect to P) and the form ϕ_- is negative definite with respect to the ordering (i.e. $-\phi_-$ is positive definite). The numbers $n_+ := \dim \phi_+$ and $n_- := \dim \phi_-$ do not change under an isometry of ϕ [KS89, Chapter 1, Section 2, Satz 2]. The integer $\operatorname{sign}_P([\phi]) := n_+ - n_-$ is defined to be *signature of* $[\phi]$ *with respect to* P. As the signature of the hyperbolic form is trivial, assigning to an isometry class $[\phi]$ its signature $\operatorname{sign}_P([\phi])$ defines a map

$$\operatorname{sign}_P : W(F) \to \mathbb{Z}$$

which is a homomorphism of rings [KS89, Chapter 1, Section 2, Satz 2]. Let $\mathrm{C}(\operatorname{sper} F, \mathbb{Z})$ denote the set of continuous integer valued functions on the real spectrum of F. The *total signature* is the ring homomorphism

$$\operatorname{sign} : W(F) \to \mathrm{C}(\operatorname{sper} F, \mathbb{Z})$$

which assigns to an isometry class $[\phi]$ the continuous function $P \mapsto \operatorname{sign}_P([\phi])$ [KS89, Chapter III, Section 8, Satz 1]. If F has no ordering, then sign is trivial.

The following lemma is obtained directly from the definition of the signature and the fact that the signature is a ring homomorphism.

Lemma 2.1. *Let P be an ordering on F.*

(1) *If ϕ is a diagonalizable form, $\phi \simeq \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$ for some $a_1, \cdots, a_n \in F^\times$, then*

$$\operatorname{sign}_P([\phi]) := \sum_{i=1}^{n} \operatorname{sgn}_P(a_i)$$

(2) *Let $a \in F^\times$. The Pfister form $\langle\langle a \rangle\rangle := \langle 1, -a \rangle$ has total signature*

$$\operatorname{sign}(\langle\langle a \rangle\rangle) = 2\mathbb{1}_{\{a<0\}}$$

(3) *Let $a_1, a_2, \cdots, a_n \in F^\times$. The n-fold Pfister form $\langle\langle a_1, \cdots, a_n \rangle\rangle := \langle\langle a_1 \rangle\rangle \otimes \cdots \otimes \langle\langle a_n \rangle\rangle$ has total signature*

$$\operatorname{sign}(\langle\langle a_1, \cdots, a_n \rangle\rangle) = 2^n \mathbb{1}_{\{a_1<0, \cdots, a_n<0\}}$$

Definition 2.4. As hyperbolic forms have even rank, assigning a quadratic form to its rank modulo 2 determines a ring homomorphism $W(F) \to \mathbb{Z}/2\mathbb{Z}$. The kernel is denoted $I(F)$ and is called the fundamental ideal of F. The powers of the fundamental ideal $I^j(F)$ are additively generated by Pfister forms $\langle\langle a_1, \cdots, a_j \rangle\rangle$, so it follows from Lemma 2.1 that the signature induces a group homomorphism

$$\operatorname{sign} : I^j(F) \to \mathrm{C}(\operatorname{sper} F, 2^j\mathbb{Z})$$

and the diagram below commutes

$$\begin{array}{ccc} I^j(F) & \xrightarrow{\text{sign}} & \mathrm{C}(\operatorname{sper} F, 2^j\mathbb{Z}) \\ \downarrow{\scriptstyle \langle\langle -1\rangle\rangle} & & \downarrow{\scriptstyle 2} \\ I^{j+1}(F) & \xrightarrow{\text{sign}} & \mathrm{C}(\operatorname{sper} F, 2^{j+1}\mathbb{Z}) \end{array}$$

So after identifying

$$\varinjlim(\mathrm{C}(\operatorname{sper} F, \mathbb{Z}) \xrightarrow{2} \mathrm{C}(\operatorname{sper} F, 2\mathbb{Z}) \xrightarrow{2} \mathrm{C}(\operatorname{sper} F, 2^2\mathbb{Z}) \xrightarrow{2} \cdots) \simeq \mathrm{C}(\operatorname{sper} F, \mathbb{Z})$$

one obtains the map

$$\varinjlim(W(F) \xrightarrow{\langle\langle -1\rangle\rangle} I(F) \xrightarrow{\langle\langle -1\rangle\rangle} I^2(F) \xrightarrow{\langle\langle -1\rangle\rangle} \cdots) \xrightarrow{\text{sign}} \mathrm{C}(\operatorname{sper} F, \mathbb{Z}) \tag{5}$$

where $\varinjlim$ denotes the colimit of the directed system of groups.

The following result first appeared in the paper of J. Arason and M. Knebusch cited in the Proposition below. Injectivity follows from A. Pfister's local-global principal [Pfi66, Satz 22] and surjectivity follows immediately from the "Normality Theorem" of R. Elman and T.Y. Lam [EL72, 3.2].

Proposition 2.5. [AK78, Satz 2a.] *The morphism* (5) *is a bijection.*

3. Residues

Throughout this section A will denote a discrete valuation ring with fraction field K and residue field $k = A/\mathbf{m}$ of characteristic different from two. Let π be a uniformizing parameter for A. The following lemma restates well-known facts on the second residue for Witt groups c.f. [MH73, Chap. IV (1.2)-(1.3)].

Lemma 3.1. (1) *Every rank one quadratic form over K is isometric to some $\langle c\rangle$, where $c = b\pi^n$, b is a unit in A, and either $n = 0$ or $n = 1$.*

(2) *The second residue $\partial_\pi : W(K) \to W(k)$ has the following description*

$$\partial_\pi(\langle c\rangle) = \begin{cases} \langle \bar{b}\rangle & \text{if } n = 1 \\ 0 & \text{if } n = 0 \end{cases}$$

on rank one forms $\langle c\rangle$ as in (1).

(3) *The second residue respects the powers of the fundamental ideal, that is, for any integer $n \geq 1$, it induces a homomorphism of groups*

$$\partial_\pi : I^n(K) \to I^{n-1}(k)$$

where $I^0(k) := W(k)$.

Definition 3.1. Let P be an ordering on the fraction field K. One says that A is *convex in K* (with respect to P) when for all $x, y, z \in K$

$$\{x \leq_P z \leq_P y \text{ and } x, y \in A\} \Rightarrow z \in A,$$

c.f. [KS89, Kapitel II, §1, Definition 1], [KS89, Kapitel II, §2, Satz 3], and [BCR98, Definition 10.1.3 (ii), Proposition 10.1.4]. If A is convex in K, then the subset $\overline{P} := \sigma(P \cap A) \subset k$, where $\sigma : A \to k$ is the surjection onto the residue field, defines an ordering on k called the *induced ordering* [KS89, Kapitel II, §2, Bemerkungen]. For any ordering $\xi \in \operatorname{sper} k$, let $Y_\xi \subset \operatorname{sper} K$ denote the subset consisting of orderings such that A is convex in K and $\xi = \overline{P}$ is the induced ordering. The assignment

$$P \mapsto \operatorname{sgn}_P(\pi)$$

defines a bijection from Y_ξ to the set $\{\pm 1\}$ [KS89, Kapitel II, §7, Theorem (Baer-Krull)], c.f. [BCR98, Theorem 10.1.10 and its proof]. That is to say, there are exactly two orderings in Y_ξ, say η_+ and η_-, where $\operatorname{sgn}_{\eta_+}(\pi) = 1$ and $\operatorname{sgn}_{\eta_-}(\pi) = -1$. The group homomorphism

$$\beta_\pi : \mathrm{C}(\operatorname{sper} K, \mathbb{Z}) \to \mathrm{C}(\operatorname{sper} A/\mathbf{m}, \mathbb{Z})$$

is defined by assigning $s \in \mathrm{C}(\operatorname{sper} K, \mathbb{Z})$ to the map $\xi \mapsto \beta_\pi(s)(\xi)$, where $\beta_\pi(s)(\xi) := s(\eta_+) - s(\eta_-)$. If $\operatorname{sper} A/\mathbf{m} = \emptyset$, then it is defined to be zero.

Lemma 3.2. *Let π be a uniformizing parameter for A. The morphism β_π of Definition (3.1) has the following description on elements* $\operatorname{sign}(\langle c\rangle)$, *where $c = b\pi^n$, b is a unit in A, and either $n = 0$ or $n = 1$:*

$$\beta_\pi(\operatorname{sign}(\langle c\rangle)) = \begin{cases} 2\operatorname{sign}(\langle \bar{b}\rangle) & \text{if } n \text{ is } 1 \\ 0 & \text{if } n \text{ is } 0. \end{cases}$$

Proof. Let $c = b\pi^n$, where b is a unit in A, and either $n = 0$ or $n = 1$. We have the following equalities which will prove the lemma. For any $\xi \in \operatorname{sper} A/\mathbf{m}$:

$$\begin{aligned}
\beta_\pi(\operatorname{sign}(\langle c\rangle))(\xi) &= \operatorname{sign}_{\eta_+}(\langle c\rangle) - \operatorname{sign}_{\eta_-}(\langle c\rangle) \\
&= \operatorname{sgn}_{\eta_+}(c) - \operatorname{sgn}_{\eta_-}(c) \\
&= \begin{cases} \operatorname{sgn}_\xi(\bar{c}) - \operatorname{sgn}_\xi(\bar{c}) & \text{if } n = 0 \text{ (both orderings induce } \xi) \\ \operatorname{sgn}_{\eta_+}(b\pi) - \operatorname{sgn}_{\eta_-}(b\pi) & \text{if } n = 1 \end{cases} \\
&= \begin{cases} 0 & \text{if } n = 0 \\ \operatorname{sgn}_{\eta_+}(b)\operatorname{sgn}_{\eta_+}(\pi) - \operatorname{sgn}_{\eta_-}(b)\operatorname{sgn}_{\eta_-}(\pi) & \text{if } n = 1 \end{cases} \\
&= \begin{cases} 0 & \text{if } n = 0 \\ \operatorname{sgn}_{\eta_+}(b) + \operatorname{sgn}_{\eta_-}(b) & \text{if } n = 1 \text{ (by definition of } \eta_+ \text{ and } \eta_-) \end{cases} \\
&= \begin{cases} 0 & \text{if } n = 0 \\ \operatorname{sgn}_\xi(\bar{b}) + \operatorname{sgn}_\xi(\bar{b}) & \text{if } n = 1 \text{ (both orderings induce } \xi) \end{cases} \\
&= \begin{cases} 0 & \text{if } n = 0 \\ 2\operatorname{sgn}_\xi(\bar{b}) & \text{if } n = 1 \end{cases}
\end{aligned}$$

□

The next lemma follows from Lemmas (3.1) and (3.2).

Lemma 3.3. *The diagram of abelian groups below is commutative.*

$$\begin{array}{ccc}
\varinjlim I^n(K) & \xrightarrow{\partial_\pi} & \varinjlim_{n \geq -1} I^n(k) \\
\Big\downarrow{\scriptstyle \text{sign}} & & \Big\downarrow{\scriptstyle 2\text{sign}} \\
\mathrm{C}(\operatorname{sper} K, \mathbb{Z}) & \xrightarrow{\beta_\pi} & \mathrm{C}(\operatorname{sper} k, \mathbb{Z})
\end{array}$$

where $\varinjlim_{n\geq -1} I^n(k)$ *denotes the colimit over*

$$W(k) \overset{\langle\langle -1\rangle\rangle}{\to} W(k) \overset{\langle\langle -1\rangle\rangle}{\to} I(k) \overset{\langle\langle -1\rangle\rangle}{\to} I^2(k) \overset{\langle\langle -1\rangle\rangle}{\to} \cdots$$

4. Real cohomology

In [Sch94], C. Scheiderer developed a theory of *real cohomology* for schemes. It "globalizes" to schemes the singular cohomology of the real points of a real variety in the same way that étale cohomology globalizes the singular cohomology of the complex points of a complex variety. We recall the definition and some properties we will need following [Sch94].

Definition 4.1. The *real spectrum of a ring* A is a topological space denoted by $\operatorname{sper} A$. As a set it consists of all pairs $\xi = (\mathbf{p}, P)$ with $\mathbf{p} \in \operatorname{spec} A$ and P an ordering of the residue field $k(\mathbf{p})$. For any point $\xi \in \operatorname{sper} A$, let $k(\xi)$ denote the real closure of the ordered field $k(\mathbf{p})$ with respect to P. For $a \in A$, write $a(\xi) > 0$ to indicate that the image of a in $k(\xi)$ is positive. The sets of the form $D(a) := \{\xi \in \operatorname{sper} A : a(\xi) > 0\}$, $a \in A$ are a subbasis for the topology on $\operatorname{sper} A$. The *real spectrum of a scheme* X is the topological space X_r formed by glueing the real spectra of its open affine subschemes. This does not depend on the open cover of X that was chosen. Furthermore, any map of schemes $f : Y \to X$ induces a continuous map of real spectra $f_r : Y_r \to X_r$. The assignment $(\mathbf{p}, P) \mapsto \mathbf{p}$ defines a continuous map of topological spaces $\operatorname{sper} A \to \operatorname{spec} A$ and similarly one has a continuous map $\operatorname{supp} : X_r \to X$ called the *support map*.

Definition 4.2. Let X be a scheme. First we recall the definition of the *real site* of X, which we will also denote by X_r. It is the category $\mathrm{O}(X_r)$ of open subsets of X_r equipped with the "usual" coverings, i.e. a family of open subspaces $\{U_\lambda \to U\}$ is a covering of $U \in \mathrm{O}(X_r)$ if $U = \cup U_\lambda$.[1] The category of sheaves of abelian groups on X_r is denoted $\mathrm{Ab}(X_r)$ and the category of abelian groups by Ab. For any $\mathcal{F} \in Ab(X_r)$, the *real cohomology groups of* X *with coefficients in* $\mathcal{F}$ are the right derived functors of the global sections functor $\Gamma : \mathrm{Ab}(X_r) \to \mathrm{Ab}$. They are denoted by

$$H^p(X_r, \mathcal{F}) := R^p\Gamma\mathcal{F}$$

where $R^p\Gamma$ is the p-th derived functor of Γ. When $X = \operatorname{spec} A$ is affine, we may write $H^p(\operatorname{sper} A, \mathcal{F})$ instead of $H^p(X_r, \mathcal{F})$. For any abelian group M, we will also denote by M the sheaf on X_r associated to the presheaf $U \mapsto M$, U any open in X_r. Such a sheaf is called a *constant sheaf.* Moreover, when the group M is equipped with the discrete topology we may write $\mathrm{C}(\operatorname{sper} A, M)$ instead of $H^0(\operatorname{sper} A, M)$. If $i : S \to X_r$ is a closed subspace, then for any abelian sheaf F on X_r, one defines

$$H^0_S(X_r, F) := \ker(F(X_r) \to F(X_r \setminus S)).$$

[1]The *real étale site*, denoted $X_{r\acute{e}t}$, is obtained by equipping the category of étale X-schemes with coverings given by the *real surjective families*, that is, $\{f_\lambda : U_\lambda \to U\}$ is a covering if the real spectrum U_r equals the union of the images $(f_\lambda)_r((U_\lambda)_r)$. For any sheaf F on X_r, $\{X' \overset{f}{\to} X\} \mapsto H^0(X'_r, f_r^*F)$ defines a sheaf on $X_{r\acute{e}t}$ denoted $F^\flat$. This determines a functor from the category $\widetilde{X_r}$ of sheaves on X_r to the category $\widetilde{X_{r\acute{e}t}}$ of sheaves on $X_{r\acute{e}t}$ which is an equivalence of categories compatible with morphisms $Y \to X$ of schemes [Sch94, Theorem 1.3, Theorem 1.14, and Remark 1.16]. We follow [Sch95, Notation] in defining real cohomology and cohomology with supports as sheaf cohomology on the topological space X_r.

The functor $F \mapsto H^0_S(X_r, F)$ is left exact and its right derived functors

$$H^q_S(X_r, F) := R^q H^0_S(X_r, F)$$

are called the *relative cohomology of* F *with support in* S [Sch95, Notations] c.f. [MR072, Exp. V, 6.3] or [Gro05, Exp. I, §2, Definition 2.1]. Additionally, $i^!F$ is defined to be the sheaf

$$S \cap U \mapsto \ker(F(U) \to F(U \setminus (S \cap U)))$$

on S (U open in X_r) and one has that

$$H^0_S(X_r, F) = H^0(X_r, i_* i^! F)$$

using the exact sequence

$$0 \to i_* i^! F \to F \to j^* j_* F \to i_* R^1 i^! F \to 0 \tag{6}$$

c.f. [MR072, Exp. V, Proposition 6.5] or [Gro05, Exp. I, Corollaire 2.11] noting that $R^1 i_* i^! F \simeq i_* R^1 i^! F$ as i_* is exact [Sch94, Corollary 3.11.1].

Remark 4.3. Let X be an algebraic variety over $\mathbb{R}$, by which we mean an $\mathbb{R}$-scheme that is separated and of finite type. We explain in this remark how to equip $X(\mathbb{R})$ with a topology and identify its singular cohomology with the real cohomology of X_r. For any affine scheme $U = \operatorname{spec} \mathbb{R}[T_1, T_2, \cdots, T_n]/I$, we consider the $\mathbb{R}$-points $U(\mathbb{R})$ as a topological space by equipping $U(\mathbb{R}) \subset \mathbb{R}^n$ with the subspace topology, where $\mathbb{R}^n$ has the Euclidean topology. The *Euclidean topology* on the set of $\mathbb{R}$-points $X(\mathbb{R})$ is the topological space formed by glueing the $U(\mathbb{R})$ of its open affine subschemes. This does not depend on the open cover of X that was chosen. The inclusion map $i : X(\mathbb{R}) \to X_r$, sending an $\mathbb{R}$-point x to the pair $(x, \mathbb{R}_{\geq 0})$, is continuous and i^{-1} induces a bijection from connected components of X_r (resp. from connected components of any basic open $D(a_1, a_2, \cdots, a_n)$ in X_r) to connected components of $X(\mathbb{R})$ (resp. to connected components of $i^{-1}(D(a_1, a_2, \cdots, a_n))$) [CR82, Corollaire 3.7]. Hence, the functor i_* determines an equivalence from the category of constant sheaves of abelian groups on $X(\mathbb{R})$ to the category of constant sheaves of abelian groups on X_r. Consequently, for any abelian group M, the sheaf cohomology $H^*(X(\mathbb{R}), M)$ coincides with the real cohomology groups $H^*(X_r, i_*M)$ and $H^*(X_r, M)$. Also, singular cohomology $H^*_{sing}(X(\mathbb{R}), M)$ is canonically isomorphic to sheaf cohomology $H^*(X(\mathbb{R}), M)$ c.f. [Sch94, Remark 13.6]. In particular, the real cohomology groups $H^*(X_r, \mathbb{Z})$ are finitely generated groups, isomorphic to $H^*_{sing}(X(\mathbb{R}), \mathbb{Z})$.

Definition 4.4. Let $\mathrm{Ab}(X_{Zar})$ denote the category of sheaves of abelian groups on the Zariski site X_{Zar}. Since the support map is a continuous map of topological spaces it induces the direct image functor

$$\mathrm{supp}_* : \mathrm{Ab}(X_r) \to \mathrm{Ab}(X_{Zar})$$

and this functor is faithful and exact [Sch94, Theorem 19.2].

Lemma 4.1. *Let* X *be a scheme. For any sheaf* $\mathcal{F} \in \mathrm{Ab}(X_r)$,

$$H^p(X_r, \mathcal{F}) \simeq H^p_{Zar}(X, \mathrm{supp}_* \mathcal{F})$$

Proof. Using the Grothendieck spectral sequence for the composition of the functors supp_* and the global sections functor Γ we obtain a spectral sequence with $\mathrm{E}_2^{p,q} = H^p_{Zar}(X, R^q \mathrm{supp}_* \mathcal{F})$ that abuts to $H^{p+q}(X_r, \mathcal{F})$. For $q > 0$, the sheaves $R^q \mathrm{supp}_* \mathcal{F}$

vanish [Sch94, Theorem 19.2]. Therefore the edge maps in this spectral sequence determine isomorphisms $H^p(X_r, \mathcal{F}) \xrightarrow{\simeq} H^p_{Zar}(X, \operatorname{supp}_* \mathcal{F})$ for $p \geq 0$. $\square$

Next we recall the work of C. Scheiderer [Sch95] in which he constructs a "Bloch-Ogus" style complex that computes real cohomology. The codimension of support filtration on X determines a spectral sequence abutting to real cohomology. Scheiderer shows that for regular excellent schemes the E_1-page is zero except for the complex $\mathrm{E}_1^{*,0}$ and hence obtains the result below. Recall that a locally noetherian scheme is called excellent if X can be covered by open affine subschemes $\operatorname{spec} A_\alpha$ where the A_α are excellent rings [Gro64, 7.8.5]. For a point $x \in X$ of a scheme, we will denote $\operatorname{sper} k(x)$ by x_r.

Proposition 4.5. [Sch95, 2.1 Theorem] *Let X be a noetherian regular excellent scheme. Let W be an open constructible subset of X_r, and let $\mathcal{F}$ be a locally constant sheaf on W. Then there is a complex of abelian groups*

$$(7) \qquad \bigoplus_{x \in X^{(0)}} H^0_x(W, \mathcal{F}) \to \bigoplus_{x \in X^{(1)}} H^1_x(W, \mathcal{F}) \to \bigoplus_{x \in X^{(2)}} H^2_x(W, \mathcal{F}) \to \cdots$$

natural in W and $\mathcal{F}$, whose qth cohomology group is canonically isomorphic to $H^q(W, \mathcal{F})$, $q \geq 0$. Here $H^q_x(W, \mathcal{F}) := H^q_{x_r \cap W}(\operatorname{sper} \mathcal{O}_{X,x} \cap W, \mathcal{F})$ are the relative cohomology groups of $\operatorname{sper} \mathcal{O}_{X,x}$ with support in $x_r \cap W$ (Definition (4.2)) and $X^{(i)}$ denotes, for $i \geq 0$, the set of codimension i points ($\dim \mathcal{O}_{X,x} = i$) of X. This complex is contravariantly functorial for flat morphisms of schemes.

The following lemma is based on the proof of [Sch95, 2.6 Proposition] where $M = \mathbb{Z}/2\mathbb{Z}$.

Lemma 4.2. *Let X be a noetherian regular excellent scheme which is integral with function field K. Let $x \in X^{(1)}$ and let π denote a choice of uniformizing parameter for $\mathcal{O}_{X,x}$. Fix an integer $n \geq 0$ and let M denote the constant sheaf $\mathbb{Z}$. Denote by ∂ the map*

$$H^0(\operatorname{sper} K, M) \to H^1_{x_r}(\operatorname{sper} \mathcal{O}_{X,x}, M)$$

induced by first differential of the complex (7) from Proposition (4.5). Then, there is an isomorphism $\iota_\pi : H^1_{x_r}(\operatorname{sper} \mathcal{O}_{X,x}, M) \to H^0(x_r, M)$ for which $\iota_\pi \circ \partial = \beta_\pi$, where β_π is the map of Definition (3.1).

Proof. Let $X' = \operatorname{sper} \mathcal{O}_{X,x}$, $Z' = x_r$, let $i : Z' \to X'$ denote the inclusion, and let $j : \operatorname{sper} K \to X'$ denote the inclusion of the complement to Z'. For any abelian sheaf M on X' the sequence

$$M \to j_* j^* M \to i_* R^1 i^!(M) \to 0$$

is exact (Definition (4.2) sequence (6)). By [Sch95, Lemma 1.3], for any locally constant sheaf M on X' the sequence

$$M \to j_* j^* M \xrightarrow{\beta} i_* i^* M \to 0$$

is exact, where β is defined on stalks as $(\beta s)_\zeta = s(\eta_+) - s(\eta_-) \in M$. Hence we get an isomorphism ι_π of cokernels and a commutative diagram

$$\begin{tikzcd} j_*j^*M(X') \arrow[r, "\partial"] \arrow[dr, "\beta"'] & i_*R^1i^!M(X') \arrow[d, "\iota_\pi"] \\ & i_*i^*M(X') \end{tikzcd} \tag{8}$$

Tracking down all the definitions, one finds that Diagram (8) is equal to the diagram below.

$$\begin{tikzcd} H^0(X' - Z', M) \arrow[r, "\partial"] \arrow[dr, "\beta_\pi"'] & H^1_{Z'}(X', M) \arrow[d, "\iota_\pi"] \\ & H^0(Z', M) \end{tikzcd}$$

where the vertical map is the isomorphism ι_π chosen, the diagonal map is the map β_π of Definition (3.1) and $\operatorname{sper} K$ equals $X' - Z'$. This finishes the proof of the lemma. □

Lemma 4.3. *Let A be a regular excellent local ring with fraction field K. Let $X = \operatorname{spec} A$, and for any $x \in X^{(1)}$, let π_x be a choice of uniformizing parameter for $\mathcal{O}_{X,x}$. Then, the sequence below is exact*

$$0 \to \mathrm{C}(\operatorname{sper} A, \mathbb{Z}) \to \mathrm{C}(\operatorname{sper} K, \mathbb{Z}) \stackrel{\oplus\beta_\pi}{\to} \bigoplus_{x\in X^{(1)}} \mathrm{C}(\operatorname{sper} k(x), \mathbb{Z})$$

where β_π is the map of Definition (3.1).

Proof. To prove the Lemma, choose isomorphisms ι_π for each $x \in X^{(1)}$ as in Lemma (4.2) and then use Proposition (4.5). □

5. On the Gersten conjecture with 2 inverted

Definition 5.1. Let A be a regular local ring with 2 invertible and let $X = \operatorname{spec} A$. Let d denote the Krull dimension of A and K the fraction field of A. We will work with the Gersten complex for the Witt groups of X as found for instance in [BGPW02, Definition 3.1], which we will denote by $C^\bullet(A, W)$. Recall that for any integer $p \geq 0$, after choosing local parameters for $\mathcal{O}_{X,x}$ for each $x \in X^{(p)}$ one may write down isomorphisms $\iota_p : C^p(A, W) \stackrel{\simeq}{\to} \bigoplus_{x\in X^{(p)}} W(k(x))$. Then, $C^\bullet(A, W)$ is isomorphic to the complex

$$C^\bullet(A, W, \iota) := W(K) \stackrel{\partial_\iota}{\to} \bigoplus_{x\in X^{(1)}} W(k(x)) \stackrel{\partial_\iota}{\to} \ldots \ldots \stackrel{\partial_\iota}{\to} \bigoplus_{x\in X^{(d)}} W(k(x))$$

where the differentials are $\partial_\iota := \iota_{p+1} \circ \partial \circ \iota_p^{-1}$ and ∂ is the differential leaving $C^p(A, W)$. The differentials ∂_ι may differ for different choices of isomorphisms ι_p but the resulting complexes will all be isomorphic. For all $x \in X^{(1)}$ we may choose parameters $\pi \in \mathcal{O}_{X,x}$ so that $\partial_\iota : W(K) \to W(k(x))$ equals the second residue ∂_π of Lemma 3.1 [BW02, Lemma 8.4] c.f. [Gil07, Proposition 6.5]. It was proved by J. Arason that the second residue ∂_π respects the filtration by powers of the fundamental ideal, that is, $\partial_\pi(I^n(K)) \subset I^{n-1}(k(x))$ [Ara75] and similarly one may show that all the differentials ∂_ι respect this filtration, for instance, this was shown by S. Gille [Gil07, Corollary 7.3] for coherent Witt groups which gives the same complex since A is regular [BGPW02, Section 3, *Another Construction*]). So one may obtain a subcomplex

$$C^\bullet(A, I^n, \iota) := \bigoplus_{x\in X^{(0)}} I^n\left(k\left(x\right)\right) \xrightarrow{\partial_\iota} \bigoplus_{x\in X^{(1)}} I^{n-1}\left(k\left(x\right)\right) \xrightarrow{\partial_\iota} \dots\dots \xrightarrow{\partial_\iota} \bigoplus_{x\in X^{(d)}} I^{n-d}\left(k\left(x\right)\right)$$

where we set $I^m\left(k\left(x\right)\right) = W\left(k\left(x\right)\right)$ when $m \leq 0$. Define

$$C^\bullet(A, W/I^n) := C^\bullet(A, W)/C^\bullet(A, I^n, \iota)$$

to be the quotient complex. The exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[r] & C^\bullet(A, I^n, \iota) \arrow[r] \arrow[d, "2"] & C^\bullet(A, W) \arrow[r] \arrow[d, "2"] & C^\bullet(A, W/I^n) \arrow[r] \arrow[d, "2"] & 0 \\ 0 \arrow[r] & C^\bullet(A, I^{n+1}, \iota) \arrow[r] & C^\bullet(A, W) \arrow[r] & C^\bullet(A, W/I^{n+1}) \arrow[r] & 0 \end{tikzcd}$$

determines a exact sequence of colimits

$$0 \longrightarrow C^\bullet(A, \varinjlim I^n) \longrightarrow C^\bullet(A, \varinjlim W) \longrightarrow C^\bullet(A, \varinjlim W/I^n) \longrightarrow 0 \tag{9}$$

where we define $C^\bullet(A, \varinjlim I^n) := \varinjlim C^\bullet(A, I^n, \iota)$, and $C^\bullet(A, \varinjlim W/I^n) := \varinjlim C^\bullet(A, W/I^n)$, and $C^\bullet(A, W[\frac{1}{2}]) := \varinjlim(C^\bullet(A, W) \xrightarrow{2} C^\bullet(A, W) \xrightarrow{2} C^\bullet(A, W) \xrightarrow{2} \cdots)$.

Theorem 5.2. *If A is a regular excellent local ring with 2 invertible, then the Gersten complex $C^\bullet(A, W[\frac{1}{2}])$ is exact and $H^0(C^\bullet(A, W[\frac{1}{2}])) = W(A)[\frac{1}{2}]$.*

Proof. We proceed by induction on the Krull dimension of A. The Gersten complex without inverting 2 is exact already in low dimensions for any regular local ring [BGPW02, Lemma 3.2]. Fix A and assume the statement of the proposition is known for regular excellent local rings of Krull dimension less than that of A. It is sufficient to show that the cohomology of $C^\bullet(A, W[\frac{1}{2}])$ vanishes in degrees 2 and higher: one may use the Balmer-Walter spectral sequence with 2 inverted for Witt groups to show that this implies $H^*(C^\bullet(A, W[\frac{1}{2}])) = 0$ in positive degree and $H^0(C^\bullet(A, W[\frac{1}{2}])) = W(A)[\frac{1}{2}]$, e.g. [BGPW02, Lemma 3.2]. For any regular parameter $f \in A$, there is a short exact sequence of complexes

$$0 \to C^\bullet(A, W) \to C^\bullet(A_f, W) \to C^\bullet(A/f, W)[-1] \to 0$$

for instance see [BGPW02, Lemma 3.3 and proof of Theorem 4.4]. Taking colimits it remains exact. As $\dim A/f$ is strictly less than $\dim A$ and A/f is again regular and excellent we have that $C^\bullet(A/f, W[\frac{1}{2}])[-1]$ is exact. Hence it remains to see that $C^\bullet(A_f, W)$ is exact in degrees 2 and higher. Note that for any $\mathbf{p} \in \operatorname{spec} A_f$, $\dim(A_f)_\mathbf{p}$ is strictly less than $\dim A$ and $(A_f)_\mathbf{p}$ is again regular and excellent, hence the cohomology of $C^\bullet(A_f, W[\frac{1}{2}])$ agrees with $H^*_{Zar}(\operatorname{spec} A_f, \varinjlim \mathcal{W})$, where $\varinjlim \mathcal{W}$ denotes the colimit over the sheaves

$$\mathcal{W} \xrightarrow{\langle\langle -1\rangle\rangle} \mathcal{W} \xrightarrow{\langle\langle -1\rangle\rangle} \mathcal{W} \xrightarrow{\langle\langle -1\rangle\rangle} \cdots$$

For any point $\mathbf{p}$ in $\operatorname{spec} A_p$, using the induction hypothesis we have that the top row in the commutative diagram below is exact and using Lemma (4.3) we have that the bottom row is exact

$$\begin{array}{ccccccc} 0 & \longrightarrow & \varinjlim W((A_f)_{\mathbf{p}}) & \longrightarrow & \varinjlim W(K) & \xrightarrow{\oplus \partial_\pi} & \bigoplus_{x \in Y^{(1)}} \varinjlim W(k(x)) \\ & & \downarrow & & \downarrow{\scriptstyle \text{sign}} & & \downarrow{\scriptstyle 2\text{sign}} \\ 0 & \longrightarrow & \mathrm{C}(\mathrm{sper}\,(A_f)_{\mathbf{p}}, \mathbb{Z}[\frac{1}{2}]) & \longrightarrow & \mathrm{C}(\mathrm{sper}\,K, \mathbb{Z}[\frac{1}{2}]) & \xrightarrow{\oplus \beta_\pi} & \bigoplus_{x \in Y^{(1)}} \mathrm{C}(\mathrm{sper}\,k(x), \mathbb{Z}[\frac{1}{2}]) \end{array}$$

where $Y := \mathrm{spec}\,(A_f)_{\mathbf{p}}$. Proposition (2.5) implies the middle vertical map is a bijection and the rightmost vertical map is an injection from which it follows that the leftmost vertical map is bijective. Thus we get an isomorphism $\varinjlim \mathcal{W} \overset{\cong}{\to} \mathrm{supp}_*\mathbb{Z}[\frac{1}{2}]$ of sheaves on A_f as it is an isomorphism on stalks, where we use Lemma (4.1) to identify the sheaf $\mathrm{supp}_*\mathbb{Z}[\frac{1}{2}]$ as the sheaf $U \mapsto \mathrm{C}(U_r, \mathbb{Z}[\frac{1}{2}])$. Then, the real cohomology groups $H^*(\mathrm{sper}\,A_f, \mathbb{Z}[1/2])$ are isomorphic to $H^*_{Zar}(\mathrm{spec}\,A_f, \varinjlim \mathcal{W})$, so it remains to prove their vanishing in degree 2 and higher. This is true since the real cohomology of local rings vanish in positive degree (in fact, semilocal too) [Sch94, Proposition (19.2.1)] and the real cohomology of sper A_f sits in a long exact sequence with that of sper A/f and sper A whenever A is regular excellent [Sch95, Corollary (1.10)]. This finishes the proof. □

Since the diagram below is commutative

$$\begin{array}{ccc} \varinjlim I^n(A) & \longrightarrow & W(A)[\frac{1}{2}] \\ \downarrow & & \downarrow \\ \varinjlim I^n(K) & \longrightarrow & W(K)[\frac{1}{2}] \end{array}$$

and the horizontal maps in the diagram are injective, we have the following Corollary to Theorem (5.2).

Corollary 5.1. *Let A be a regular excellent local ring with 2 invertible. The map*

$$\varinjlim I^n(A) \to \varinjlim I^n(K)$$

is injective.

We will also need the following result later.

Lemma 5.1. *Let A be a regular excellent local ring with 2 invertible. The cohomology groups $H^m(C^\bullet(A, \varinjlim I^n))$ vanish when $m \geq 2$.*

Proof. Consider the long exact sequence in cohomology

$$\cdots \to H^m(C^\bullet(A, \varinjlim I^n)) \to H^m(C^\bullet(A, \varinjlim W)) \to H^m(C^\bullet(A, \varinjlim W/I^n)) \to \cdots$$

associated to the short exact sequence of complexes (9). The cohomology groups $H^m(C^\bullet(A, \varinjlim W))$ vanish when $m > 0$ by Theorem (5.2). Then $H^m(C^\bullet(A, \varinjlim I^n))$ is isomorphic to $H^{m-1}(C^\bullet(A, \varinjlim W/I^n))$ for all $m \geq 2$. The cohomology groups $H^m(C^\bullet(A, \varinjlim W/I^n))$ are two-primary torsion since the complex $C^\bullet(A, \varinjlim W/I^n)$ is, while the groups $H^m(C^\bullet(A, \varinjlim I^n))$ have no 2-primary torsion since multiplication by 2

$$C^\bullet(A, \varinjlim I^n) \xrightarrow{2} C^\bullet(A, \varinjlim I^n)$$

is an isomorphism of complexes. Thus both groups vanish proving the lemma. □

6. Purity of the limit in the local "geometric" case

For any prime p, we will use $\mathbb{Z}_{\langle p\rangle}$ to denote the localization of $\mathbb{Z}$ at the prime ideal $\langle p\rangle \in \operatorname{spec}\mathbb{Z}$. In this section we prove purity of $\varinjlim I^n(A)$ in the case that A is essentially smooth over either $\mathbb{Q}$ or $\mathbb{Z}_{\langle p\rangle}$ (Proposition (6.1)). When A is a local ring of mixed-characteristic $(0,p)$ with $p \neq 2$ (that is to say, the characteristic of the fraction field K is 0 and the characteristic of the residue field is p) we will say that A is essentially smooth over $\mathbb{Z}_{\langle p\rangle}$ if $A = R_{\mathbf{p}}$ is the localization at a prime $\mathbf{p} \in \operatorname{spec} R$ of a smooth and finite type $\mathbb{Z}_{\langle p\rangle}$-algebra $R = \mathbb{Z}_{\langle p\rangle}[T_1, T_2, \cdots, T_n]/I$.

Lemma 6.1. *If A is essentially smooth over $\mathbb{Z}_{\langle p\rangle}$ for some prime $p \neq 2$ or over $\mathbb{Q}$, then the sequence*

$$I^n(A)/I^{n+1}(A) \to I^n(K)/I^{n+1}(K) \overset{\oplus\partial_\pi}{\to} \bigoplus_{x\in X^{(1)}} I^{n-1}(k(x))/I^n(k(x))$$

is exact, where $X = \operatorname{spec} A$ and K is the fraction field of A.

Proof. Let $K_n^M(A)/2$ denote the "naive" Milnor K-theory defined exactly as for a field. Kummer theory gives a "symbol map" $K_n^M(A)/2 \to H^n_{ét}(A,\mathbb{Z}/2)$ and in the commutative diagram

$$\begin{tikzcd}
 & K_n^M(A)/2 \arrow[r] \arrow[d] & K_n^M(K)/2 \arrow[r] \arrow[d] & \bigoplus_{x\in X^{(1)}} K_{n-1}^M(k(x))/2 \arrow[d] \\
0 \arrow[r] & H^n_{ét}(A,\mathbb{Z}/2) \arrow[r] & H^n_{ét}(K,\mathbb{Z}/2) \arrow[r] & \bigoplus_{x\in X^{(1)}} H^{n-1}_{ét}(k(x),\mathbb{Z}/2)
\end{tikzcd}$$

where $X = \operatorname{spec} A$ and K is the fraction field of A, the lower row is exact as a consequence of Gillet's Gersten conjecture for étale cohomology in the $\mathbb{Z}_{\langle p\rangle}$ case[2], and Bloch-Ogus in the $\mathbb{Q}$-case. Furthermore, the Galois symbol

$$K_n^M(A)/2 \to H^n_{ét}(A,\mathbb{Z}/2)$$

is surjective when A is essentially smooth over $\mathbb{Q}$ [Ker09, Ker10] and when A is essentially smooth over a discrete valuation ring[3] c.f. [Kah02, p.114, surjectivity of the Galois symbol]. Applying the Milnor conjecture as proved by V. Voevodsky, we have that the vertical maps in the middle and on the right are bijections. It follows that the upper row is exact in the middle. Since $\langle\langle a, 1-a\rangle\rangle = 0$ in $W(A)$ for $a \in A^\times$ such that $1-a \in A^\times$, there is a well-defined homomorphism $K_n^M(A)/2 \to I^n(A)/I^{n+1}(A)$. Hence, in the commutative diagram

$$\begin{tikzcd}
K_n^M(A)/2 \arrow[r] \arrow[d] & K_n^M(K)/2 \arrow[r] \arrow[d] & \bigoplus_{x\in X^{(1)}} K_{n-1}^M(k(x))/2 \arrow[d] \\
I^n(A)/I^{n+1}(A) \arrow[r] & I^n(K)/I^{n+1}(K) \arrow[r] & \bigoplus_{x\in X^{(1)}} I^{n-1}(k(x))/I^n(k(x))
\end{tikzcd}$$

[2] Manuscript notes titled "Bloch-Ogus for the étale cohomology of certain arithmetic schemes" distributed at the 1997 Seattle algebraic K-theory conference. Also, this follows from Thomas Geisser's proof of the Gersten conjecture for motivic cohomology [Gei04, This is explicitly stated in the sentence after Theorem 1.2, because $R^n\epsilon_*\mu_2$ is the Zariski sheaf associated to the presheaf $U \mapsto H^n_{ét}(U,\mu_2)$ and the affirmation of the Milnor conjecture allows one to identify the Gersten complex for motivic cohomology with the Gersten complex for étale cohomology.]

[3] In a correspondence with the author, B. Kahn explained that the passage from surjectivity in the essentially smooth over a field case to this case is easy and goes back to Lichtenbaum, if you grant Gillet's Gersten conjecture for étale cohomology.

after using again the Milnor conjecture, by which the vertical maps in the middle and on the right are bijections, we have that the lower row is exact in the middle. $\square$

Lemma 6.2. *Let A be essentially smooth over $\mathbb{Z}_{\langle p \rangle}$, $p \neq 2$, or $\mathbb{Q}$.*

(1) *There exists an integer N such that $C^\bullet(A, I^s, \iota) \xrightarrow{2} C^\bullet(A, I^{s+1}, \iota)$ is an isomorphism of complexes for all $s \geq N$.*
(2) *The groups $H^m(C^\bullet(A, W))$ are 2^N-torsion for all $m \geq 2$.*
(3) *There exists an integer $B \geq 0$ such that $2^B H^0(C^\bullet(A, W)) \subset i_*(W(A))$, where $i^* : W(A) \to W(K)$ denotes the map induced by $i : \operatorname{spec} K \to \operatorname{spec} A$.*
(4) $2^{B+N} H^0(C^\bullet(A, W)) \subset i_*(I^N(A))$

Proof. To prove (1), note that the cohomological 2-dimension of $k(x)[\sqrt{-1}]$ is finite and for all points x, bounded, strictly less than some integer n. Using the Arason-Pfister Haupsatz and the Milnor conjecture for fields it follows that $I^n(k(x)[\sqrt{-1}])$ vanishes for all x, and from this it follows that, for all x, we have an isomorphism $I^s(k(x)) \xrightarrow{2} I^{s+1}(k(x))$ for all $s \geq n$ [EKM08, Corollary 35.27]. Hence $C^\bullet(A, I^s, \iota) \xrightarrow{2} C^\bullet(A, I^{s+1}, \iota)$ is an isomorphism of complexes for all $s \geq N$, where $N := n + \dim X$. Then $C^\bullet(A, \varinjlim I^n)$ and $C^\bullet(A, I^N, \iota)$ are isomorphic complexes, so the cohomology group $H^m(C^\bullet(A, I^N, \iota))$ vanishes when $m \geq 2$ by Lemma 5.1. It follows that the groups $H^m(C^\bullet(A, W))$ are 2^N-torsion when $m \geq 2$ since $H^m(C^\bullet(A, W)) \xrightarrow{2^N} H^m(C^\bullet(A, W))$ factors

$$\begin{tikzcd} H^m(C^\bullet(A,W)) \arrow[rr, "2^N"] \arrow[dr, "2^N"'] & & H^m(C^\bullet(A,W)) \\ & H^m(C^\bullet(A, I^N, \iota)) \arrow[ur] & \end{tikzcd}$$

proving (2). Now to prove (3), let $q \in H^0(C^\bullet(A, W))$. From the Balmer-Walter spectral sequence for Witt groups [BW02] we have that $W(A)$ surjects onto $E_\infty^{0,0}$ which consists of the elements in $H^0(C^\bullet(A, W))$ mapped to zero under all the differentials in the spectral sequence leaving $H^0(C^\bullet(A, W))$. So it suffices to show that some 2-power of q maps to zero under all of these finitely many non-trivial differentials. The first non-trivial differential is $d : H^0(C^\bullet(A, W)) \to H^5(C^\bullet(A, W))$. Since $2^N H^5(C^\bullet(A, W)) = 0$, we have that $d(2^N q) = 0$. Repeating this argument for each non-trivial differential $d : H^0(C^\bullet(A, W)) \to H^{4*+1}(C^\bullet(A, W))$ we eventually find some 2-power 2^B, which does not depend on q, such that $2^B q$ is in the kernel of all differentials, hence is in $E_\infty^{0,0}$. Finally, to prove (4), let $q \in 2^{B+N} H^0(C^\bullet(A, W))$. Write it as $q = 2^{B+N} q_{unr}$ for some $q_{unr} \in H^0(C^\bullet(A, W))$. By (3) we have that $2^B q_{unr} = i_*(Q)$ for some $Q \in W(A)$. So $i_*(2^N Q) = q$ and $2^N Q \in I^N(A)$. This proves $2^{B+N} H^0(C^\bullet(A, W)) \subset i_*(I^N(A))$, finishing the proof of the Lemma. $\square$

Proposition 6.1. *Let A be essentially smooth over either $\mathbb{Z}_{\langle p \rangle}$, $p \neq 2$, or $\mathbb{Q}$. The sequence*

$$\varinjlim I^n(A) \to \varinjlim I^n(K) \xrightarrow{\oplus \partial_\pi} \bigoplus_{x \in X^{(1)}} \varinjlim_{n \geq -1} I^n(k(x))$$

is exact, where $\varinjlim_{n \geq -1} I^n(k(x))$ denotes the colimit over

$$W(k(x)) \xrightarrow{\langle\langle -1 \rangle\rangle} W(k(x)) \xrightarrow{\langle\langle -1 \rangle\rangle} I(k(x)) \xrightarrow{\langle\langle -1 \rangle\rangle} I^2(k(x)) \xrightarrow{\langle\langle -1 \rangle\rangle} \cdots$$

.

Proof. Let q in the kernel of the residue, hence $q \in H^0(C^\bullet(A, I^N, \iota))$ for some $N \geq 0$. We may assume that N is the integer N from Lemma 6.2 (1) by either multiplying by 2 or dividing by 2 as needed. Using Lemma 6.1 we find $\overline{Q_N} \in I^N(A)/I^{N+1}(A)$ which we may then lift to obtain a $Q_N \in I^N(A)$ satisfying $q - i_*(Q_N) \in H^0(C^\bullet(A, I^{N+1}, \iota))$. By repeating this argument we find that $q - i_*(Q_N + Q_{N+1} + \cdots + Q_{B+2N-1}) \in H^0(C^\bullet(A, I^{B+2N}, \iota))$ where B is the integer from Lemma 6.2 (3). Since we are in the "stable" range we have that $H^0(C^\bullet(A, I^{B+2N}, \iota)) = 2^{B+N} H^0(C^\bullet(A, I^N, \iota)) \subset 2^{B+N} H^0(C^\bullet(A, W)) \subset i_*(I^N(A))$, where we used Lemma 6.2 (4) to obtain the rightmost inclusion. Hence we have a $Q_N^{'} \in I^N(A)$ such that

$$q = i_*(Q_N + Q_{N+1} + \cdots + Q_{B+2N-1} + Q_N^{'})$$

where $Q_N + Q_{N+1} + \cdots + Q_{B+2N-1} + Q_N^{'} \in I^N(A)$. This finishes the proof. $\square$

7. On the signature: Local case

In this section we use "Hoobler's trick" which is a method due to R. Hoobler [Hoo06] for passing from the smooth geometric case to the geometric case for many questions involving cohomological invariants satisfying "rigidity" in the sense of the following Lemma.

Lemma 7.1. *If B is a local ring and (B, I) a henselian pair such that 2 is invertible in both B and B/I, then for all integers $n \geq 0$, the homomorphisms of groups*

$$I^n(B) \to I^n(B/I)$$

and

$$I^n(B)/I^{n+1}(B) \to I^n(B/I)/I^{n+1}(B/I)$$

induced by the surjection $B \to B/I$ are bijections.

Proof. Let B a local ring and (B, I) a henselian pair such that 2 is invertible in both B and B/I. Considering the diagram

$$\begin{tikzcd}
0 \arrow[r] & I^{n+1}(B) \arrow[r] \arrow[d] & I^n(B) \arrow[r] \arrow[d] & I^n(B)/I^{n+1}(B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^{n+1}(B/I) \arrow[r] & I^n(B/I) \arrow[r] & I^n(B/I)/I^{n+1}(B/I) \arrow[r] & 0
\end{tikzcd}$$

we see that, by the two out of three lemma, it is sufficient to prove that $I^n(B) \to I^n(B/I)$ is a bijection for all $n \geq 0$. To prove injectivity for all $n \geq 0$, note that as $I^n(B)$ is contained in $W(B)$, it suffices to prove that $W(B) \to W(B/I)$ is injective.

We claim that the assignment $b+I \mapsto b$ determines a well-defined map $(B/I)^\times/(B/I)^{\times 2} \to B^\times/B^{\times 2}$. This claim follows from rigidity for étale cohomology due to Strano and Gabber independently but one may also prove it directly from the definition of Henselian pair [4]: let $b_1, b_2 \in B^\times$ such that $b_1 + I = b_2 + I$; the polynomial $T^2 - \frac{b_1}{b_2}$ has image $T^2 - 1$ in $B/I[T]$; as (B, I) is a henselian pair, from the factorization $T^2 - 1 = (T-1)(T+1)$ in $B/I[T]$ we obtain a factorization $T^2 - \frac{b_1}{b_2} = (T-a)(T+a)$ in $B[T]$, for some $a \in B$; hence $b_1 = a^2 b_2$ for some $a \in B^\times$, that is, $b_1 = b_2$ in $B^\times/(B^\times)^2$. The claim follows. Next recall that for any semilocal ring A, the Witt

[4]The author learned this from a recent preprint of Stefan Gille titled *On quadratic forms over semilocal rings*

group $W(A)$ is a quotient of the group ring $\mathbb{Z}[A^\times/{A^\times}^2]$ modulo the set of relations R additively generated by $[1]+[-1]$ and all elements

$$\sum_{i=1}^{h}[a_i] - \sum_{i=1}^{h}[b_i]$$

satisfying

$$\perp_{i=1}^{h} \langle a_i\rangle \simeq \perp_{i=1}^{h} \langle b_i\rangle$$

with $h = 4$ [Kne77, Ch. 2, §4, Theorem 2]. Hence, in the commutative diagram below

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] & \mathbb{Z}[B^\times/{B^\times}^2] \arrow[r] & W(B) \arrow[r] & 0 \\
0 \arrow[r] & R \arrow[r] \arrow[u] & \mathbb{Z}[(B/I)^\times/{(B/I)^\times}^2] \arrow[r] \arrow[u] & W(B/I) \arrow[r] & 0
\end{tikzcd}$$

the rows are exact. Thus we obtain a well-defined map of cokernels $W(B/I) \to W(B)$ such that the composition $W(B) \to W(B/I) \to W(B)$ is the identity. This proves the desired injectivity. The composition $W(B/I) \to W(B) \to W(B/I)$ is the identity hence $W(B) \to W(B/I)$ is surjective. To prove surjectivity of $I^n(B) \to I^n(B/I)$ for all $n \geq 0$, recall that $I^n(B/I)$ is additively generated by Pfister forms $\langle\langle \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n\rangle\rangle$ where $\bar{b}_1, \bar{b}_2, \ldots \bar{b}_n$ are units in B/I [Bae78, Ch. V, Section 1, Remark 1.3]. For any Pfister form $\langle\langle \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n\rangle\rangle$ we may lift the $\bar{b}_i$ to units b_i of B to obtain an element $\langle\langle b_1, b_2, \ldots, b_n\rangle\rangle \in I^n(B)$ mapping to it, proving surjectivity of $I^n(B) \to I^n(B/I)$ and finishing the proof of the lemma. ☐

Proposition 7.1. *If A is a local ring with $2 \in A^\times$, then the signature map*

$$\varinjlim I^n(A) \to \mathrm{C}(\operatorname{sper} A, \mathbb{Z})$$

is a bijection.

Proof. As both groups respect filtered colimits, it suffices to consider the case where A is a localization of a finite type $\mathbb{Z}$-algebra: any local ring may be written as a union of its finitely generated subrings A_α; pulling back the maximal ideal of A over $A_\alpha \to A$ yields a prime ideal $\mathbf{p}_\alpha \in \operatorname{spec} A_\alpha$; localizing the A_α with respect to these primes yields a directed system of local rings $A_{\mathbf{p}_\alpha}$ and taking the direct limit yields A. From now on we assume $A = R_{\mathbf{p}}$, where $\mathbf{p} \in \operatorname{spec} R$ and $R = \mathbb{Z}[T_1, T_2, \cdots, T_n]/I$ for some ideal I. We obtain a *henselian pair* (B, I) *for* A as follows: let s denote the quotient map $\mathbb{Z}[T_1, T_2, \cdots, T_n] \to R$ and let $B_0 := \mathbb{Z}[T_1, T_2, \cdots, T_n]_{s^{-1}(\mathbf{p})}$ and similarly $I_0 := I_{s^{-1}(\mathbf{p})}$; let B denote the henselization of B_0 along I_0 and $I := I_0B$. Recall, the henselization along I_0 is obtained by taking the colimit over the directed category consisting of those étale B_0-algebras C having the property that $B_0/I_0 \to C/I_0C$ is an isomorphism. The map $B_0 \to B$ induces on quotients $A = B_0/I_0 \to B/I$ an isomorphism of local rings. In the commutative diagram below, the horizontal maps induced by the surjection $B \to B/I \simeq A$

$$\begin{tikzcd}
\varinjlim I^n(B) \arrow[r] \arrow[d, "\mathrm{sign}"] & \varinjlim I^n(A) \arrow[d, "\mathrm{sign}"] \\
\mathrm{C}(\operatorname{sper} B, \mathbb{Z}) \arrow[r] & \mathrm{C}(\operatorname{sper} A, \mathbb{Z})
\end{tikzcd}$$

are isomorphisms for the powers of the fundamental ideal (Lemma (7.1)) and for real cohomology[5]. Therefore it suffices to prove bijectivity for B. We claim that the local ring B is a filtered colimit of local rings which are essentially smooth over either $\mathbb{Z}_{\langle p\rangle}$, $p \neq 2$, or over $\mathbb{Q}$. In order to prove the claim, first note that the pullback of $s^{-1}(\mathbf{p}) \in \operatorname{spec}\mathbb{Z}[T_1, T_2, \cdots, T_n]$ over $\mathbb{Z} \to \mathbb{Z}[T_1, T_2, \cdots, T_n]$ yields a prime $\langle p\rangle \in \operatorname{spec}\mathbb{Z}$ and localizing with respect to this prime induces $\mathbb{Z}_{\langle p\rangle} \hookrightarrow B_0$. When $\langle p\rangle = 0$ it follows that B_0 contains $\mathbb{Q}$, otherwise B_0 contains $\mathbb{Z}_{\langle p\rangle}$, $p \neq 2$. The morphisms $\mathbb{Z}_{\langle p\rangle} \to B_0$ and $B_0 \to B$ are both flat with geometrically regular fibers, hence the composition $\mathbb{Z}_{\langle p\rangle} \to B$ has these properties. Then it follows from Popescu's theorem that B is a filtered colimit of either smooth $\mathbb{Z}_{\langle p\rangle}$-algebras or $\mathbb{Q}$-algebras A_α. Pulling back the maximal ideal over $A_\alpha \to B$ and localizing one obtains the statement of the claim. Thus, we may assume that B is essentially smooth over $\mathbb{Q}$ or $\mathbb{Z}_{\langle p\rangle}$. Then we map apply Lemma (4.3) to get exactness of the lower row in the commutative diagram below

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim I^n(B) \arrow[r] \arrow[d] & \varinjlim I^n\,(K) \arrow[r, "\oplus\partial_\pi"] \arrow[d, "\mathrm{sign}"] & \bigoplus_{x \in Y^{(1)}} \varinjlim_{n \geq -1} I^n(k(x)) \arrow[d, "2\mathrm{sign}"] \\
0 \arrow[r] & \mathrm{C}(\operatorname{sper} B, \mathbb{Z}) \arrow[r] & \mathrm{C}(\operatorname{sper} K, \mathbb{Z}) \arrow[r, "\oplus\beta_\pi"] & \bigoplus_{x \in Y^{(1)}} \mathrm{C}(\operatorname{sper} k(x), \mathbb{Z})
\end{tikzcd}$$

where $Y = \operatorname{spec} B$. We have exactness of the upper row by Proposition (6.1) and Corollary (5.1). Using the bijection of Proposition (2.5) we get that the middle vertical map in the diagram above is bijective and the rightmost vertical map is injective. The square on the right commutes by Lemma (3.3), hence $\varinjlim I^n(B) \to \mathrm{C}(\operatorname{sper} B, \mathbb{Z})$ is bijective, finishing the proof of the theorem. $\square$

The following corollary is well-known as mentioned in the introduction.

Corollary 7.1. *Let A be a local ring with $2 \in A^\times$. Then the signature induces an isomorphism*

$$W(A)[\frac{1}{2}] \to \mathrm{C}(\operatorname{sper} A, \mathbb{Z})[\frac{1}{2}]$$

Proof. From the preceding theorem we have that any $f \in \mathrm{C}(\operatorname{sper} A, \mathbb{Z})$ has $2^n f = \operatorname{sign}(Q)$ for some $Q \in I^n(A) \subset W(A)$, proving surjectivity, and that for any $Q^{'} \in W(A)$, if $\operatorname{sign}(Q^{'}) = 0$ then $2^n Q^{'} = 0$ for some n, proving injectivity. $\square$

Remark 7.2. Let $A = \bigoplus_{n \geq 0} A_n$ be a $\mathbb{Z}_+$-graded ring and let $s \in A_1$ be a homogeneous element of degree 1. Recall that the homogeneous localization $A_{(s)}$ is the subring of degree zero elements in the localization of A with respect to $\{1, s, s^2, \cdots\}$, and that $A_{(s)} \simeq A/(s-1)A$ as rings. Furthermore, $A_{(s)}$, may be obtained by taking the direct limit of the sequence $A_0 \xrightarrow{s} A_1 \xrightarrow{s} A_2 \xrightarrow{s} \cdots$.

Corollary 7.2. *Let A be a local ring with 2 invertible.*

[5]The following proof was communicated to the author by C. Scheiderer: every point in $\operatorname{sper} B$ specializes to a point in $\operatorname{sper} B/I$ by the henselian property; since any real spectrum is a "normal" spectral space, meaning that every point of X_r specializes to a unique closed point, the restriction map in sheaf cohomology $H^*(X_r, \mathcal{F}) \to H^*((X_r)_{max}, \mathcal{F})$ is an isomorphism for every sheaf $\mathcal{F}$ on any scheme X; thus restriction gives isomorphisms $H^*(\operatorname{sper}(B), \mathcal{F}) \to H^*(\operatorname{sper}(B)_{max}, \mathcal{F}) \overset{\simeq}{\leftarrow} H^*(\operatorname{sper}(B/I), \mathcal{F})$.

(1) *Let* $I^*(A)_{\langle\langle -1\rangle\rangle}$ *denote the homogeneous localization of the graded ring* $\bigoplus_{n\geq 0} I^n(A)$ *with respect to the element* $\langle\langle -1\rangle\rangle = \langle 1,1\rangle \in I(A)$*. The signature defines an isomorphism of rings*

$$I^*(A)_{\langle\langle -1\rangle\rangle} \simeq \mathrm{C}(\operatorname{sper} A, \mathbb{Z})$$

(2) *Let* $\overline{I^*}(A)_{\langle\langle -1\rangle\rangle}$ *denote the homogeneous localization of the graded ring* $\bigoplus_{n\geq 0} \overline{I^n}(A)$ *with respect to* $\langle\langle -1\rangle\rangle = \langle 1,1\rangle \in \overline{I^1}(A)$*, where* $\overline{I^n}(A) := I^n(A)/I^{n+1}(A)$*. The signature defines an isomorphism of rings*

$$\overline{I^*}(A)_{\langle\langle -1\rangle\rangle} \simeq \mathrm{C}(\operatorname{sper} A, \mathbb{Z}/2)$$

Proof. Recall (Remark 7.2) that one may identify $\varinjlim I^n(A)$ with $I^*(A)_{\langle\langle -1\rangle\rangle}$: Using the direct sum construction of the direct limit $\varinjlim I^n(A)$ the relations one finds are the same as the relations defining the localization $I^*(A)_{\langle\langle -1\rangle\rangle}$; Explicitly, the isomorphism $\varphi : \varinjlim I^n(A) \to I^*(A)_{\langle\langle -1\rangle\rangle}$ is given by $\varphi_n : I^n(A) \to I^*(A)_{\langle\langle -1\rangle\rangle}$ defined by

$$q \mapsto \frac{q}{\langle\langle -1\rangle\rangle^n}$$

and consequently we obtain using the preceding proposition that the assignment

$$\frac{q}{\langle\langle -1\rangle\rangle^n} \mapsto \frac{\operatorname{sign}(q)}{2^n}$$

where $q \in I^n(A)$ defines an isomorphism from $I^*(A)_{\langle\langle -1\rangle\rangle}$ to $\mathrm{C}(\operatorname{sper} A, \mathbb{Z})$. To prove (2), we obtain the desired isomorphism as an isomorphism of cokernels in the commutative diagram below

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim_{n\geq 1} I^n(A) \arrow[r] \arrow[d] & \varinjlim I^n(A) \arrow[r] \arrow[d] & \varinjlim \overline{I^n}(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{C}(\operatorname{sper} A, 2\mathbb{Z}) \arrow[r] & \mathrm{C}(\operatorname{sper} A, \mathbb{Z}) \arrow[r] & \mathrm{C}(\operatorname{sper} A, \mathbb{Z}/2) \arrow[r] & 0
\end{tikzcd}$$

where $\varinjlim_{n\geq 1} \to \mathrm{C}(\operatorname{sper} A, 2\mathbb{Z})$ is an isomorphism since in the commutative diagram

$$\begin{tikzcd}
\varinjlim_{n\geq 1} I^n(A) \arrow[r] & \mathrm{C}(\operatorname{sper} A, 2\mathbb{Z}) \\
\varinjlim I^n(A) \arrow[r] \arrow[u, "\langle\langle -1\rangle\rangle"] & \mathrm{C}(\operatorname{sper} A, \mathbb{Z}) \arrow[u, "2"']
\end{tikzcd}$$

the vertical maps are isomorphisms as is the lower horizontal map. □

Corollary 7.3. *Let A be a local ring with 2 invertible. Let* $H^*_{ét}(A, \mathbb{Z}/2)_{(-1)}$ *denote the homogeneous localization of the cohomology ring* $\bigoplus_{n\geq 0} H^n_{ét}(A, \mathbb{Z}/2\mathbb{Z})$ *with respect to* $(-1) \in H^1_{ét}(A, \mathbb{Z}/2\mathbb{Z})$*. Then the nth cohomological invariant* $\overline{e}_n : \overline{I^n} \to H^n_{ét}(A, \mathbb{Z}/2)$*, which assigns the class of a Pfister form* $\langle\langle a_1, \cdots, a_n\rangle\rangle$ *to the cup product* $(a_1) \cup \cdots \cup (a_n)$ *determines a well-defined homomorphism*

$$\overline{e}_* : \overline{I^*}(A)_{\langle\langle -1\rangle\rangle} \simeq H^*_{ét}(A, \mathbb{Z}/2)_{(-1)}$$

which is an isomorphism of rings.

Proof. For any local ring A essentially smooth over $\mathbb{Z}_{\langle p\rangle}$ or $\mathbb{Q}$, the diagram commutes

$$\begin{tikzcd}
 & I^n(A) \arrow[r] \arrow[d] & I^n(K) \arrow[r] \arrow[d] & \bigoplus_{x\in X^{(1)}} I^{n-1}(k(x))/2 \arrow[d] \\
0 \arrow[r] & H^n_{\acute{e}t}(A,\mathbb{Z}/2) \arrow[r] & H^n_{\acute{e}t}(K) \arrow[r] & \bigoplus_{x\in X^{(1)}} H^{n-1}_{\acute{e}t}(k(x))
\end{tikzcd}$$

and the lower row is exact as the Gersten conjecture is known for étale cohomology in this case. As the diagram commutes it follows that $I^n(A)/I^{n+1}(A)$ maps into $H^n(A,\mathbb{Z}/2)$. Let $\overline{e}_n$ denote this map. As the lower row is exact, it has the description asserted on Pfister forms. Using rigidity and the fact that both groups respect filtered colimits as we did in the proof of Theorem (8.5) we obtain the map $\overline{e}_n$ for any local ring, and after localizing, we obtain the map in the commutative diagram below

$$\begin{tikzcd}
\overline{I^*}(A)_{\langle\langle -1\rangle\rangle} \arrow[r, "\overline{e}_*"] \arrow[dr, "\simeq"] & H^*_{\acute{e}t}(A,\mathbb{Z}/2)_{(-1)} \arrow[d, "\simeq"] \\
 & \mathrm{C}(\operatorname{sper} A,\mathbb{Z}/2)
\end{tikzcd}$$

where we use that for any semi-local ring A with 2 invertible the signature modulo 2 defines an isomorphism

$$H^*_{\acute{e}t}(A,\mathbb{Z}/2)_{(-1)} \xrightarrow{\simeq} \mathrm{C}(\operatorname{sper} A,\mathbb{Z}/2) \tag{10}$$

of rings. This is due to J. Burési and L. Mahé in the semi-local case [Mah95, Bur95] and C. Scheiderer in general [Sch94, Corollary (7.10.3) and (7.19)]. From the isomorphisms in the diagram, the desired isomorphism follows. $\square$

8. Globalization

In this section X will always denote a scheme. Let $W(X)$ denote the Witt ring of symmetric bilinear forms over X c.f. [Kne77].

Definition 8.1. Recall that the *global signature* is the ring homomorphism

$$\operatorname{sign} : W(X) \to H^0(X_r,\mathbb{Z})$$

that assigns an isometry class $[\phi]$ of a symmetric bilinear form ϕ over X to the function on X_r defined by

$$\operatorname{sign}([\phi])(x,P) := \operatorname{sign}_P([i_x^*\phi])$$

where $i_x : x \to X$ is any point and P is any ordering on $k(x)$ c.f. [Mah82].

Definition 8.2. There is a well-defined ring homomorphism $W(X) \to H^0_{\acute{e}t}(X,\mathbb{Z}/2\mathbb{Z})$, called the *rank*, which assigns an isometry class of a symmetric bilinear form $[\mathcal{E},\phi]$ over X to the rank of its underlying vector bundle $\mathcal{E}$ modulo 2 c.f. [Kne77, Chapter 1, §7]. The kernel of the rank map is called the *fundamental ideal* and is denoted by $I(X)$.

It follows from the definitions that the diagram below commutes.

$$
\begin{tikzcd}
W(X) \arrow[r, "\mathrm{sign}"] \arrow[d, "\text{Rank mod 2}"] & H^0(X_r,\mathbb{Z}) \arrow[d] \\
H^0_{ét}(X,\mathbb{Z}/2\mathbb{Z}) \arrow[r, "h_0"] & H^0(X_r,\mathbb{Z}/2\mathbb{Z})
\end{tikzcd} \tag{11}
$$

where h_0 denotes the *signature modulo 2* defined as follows: if $\alpha \in H^0_{ét}(X,\mathbb{Z}/2\mathbb{Z})$ and $\xi : x \to X$ is the inclusion of a "real" point, that is, for some $(x,P) \in X_r$, then $h_0(\alpha)$ evaluated at ξ is $\xi^*\alpha \in H^0(x_{ét},\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$; write $\alpha(\xi)$ for this element of $\mathbb{Z}/2\mathbb{Z}$, so $h_0(\alpha)$ is the locally constant map $X_r \to \mathbb{Z}/2\mathbb{Z}$, $\xi \mapsto \alpha(\xi)$ c.f. [Sch94, (7.19.1)].

Definition 8.3. As there is an exact sequence

$$0 \to H^0(X_r,2\mathbb{Z}) \to H^0(X_r,\mathbb{Z}) \to H^0(X_r,\mathbb{Z}/2\mathbb{Z})$$

one finds using commutativity of Diagram (11) that the restriction of the signature to $I(X)$ defines the homomorphism of groups below.

$$I(X) \to H^0(X_r,2\mathbb{Z})$$

For $n \geq 0$, let $I^n(X)$ denote the powers of the fundamental ideal and $I^0(X) = W(X)$. Since the global signature is a ring homomorphism that maps elements of $I(X)$ into $H^0(X_r,2\mathbb{Z})$, it follows that for any $n \geq 0$ it induces a homomorphism

$$I^n(X) \to H^0(X_r,2^n\mathbb{Z})$$

of groups. Moreover, multiplication by $2 = \langle\langle -1\rangle\rangle \in I(X)$ induces a homomorphism $I^n(X) \overset{\langle\langle -1\rangle\rangle}{\to} I^{n+1}(X)$ such that the diagram below commutes.

$$
\begin{tikzcd}
I^j(X) \arrow[r, "\mathrm{sign}"] \arrow[d, "\langle\langle -1\rangle\rangle"] & H^0(X_r,2^j\mathbb{Z}) \arrow[d, "2"] \\
I^{j+1}(X) \arrow[r, "\mathrm{sign}"] & H^0(X_r,2^{j+1}\mathbb{Z})
\end{tikzcd}
$$

Hence, we obtain a homomorphism

$$\varinjlim I^n(X) \to H^0(X_r,\mathbb{Z})$$

where $\varinjlim I^n(X)$ denotes the direct limit of the sequence $W(X) \overset{\langle\langle -1\rangle\rangle}{\to} I(X) \overset{\langle\langle -1\rangle\rangle}{\to} I^2(X) \overset{\langle\langle -1\rangle\rangle}{\to} \cdots$ of groups.

Definition 8.4. It follows from Lemma (4.1) that $\mathrm{supp}_* \mathbb{Z}$ is the Zariski sheaf $U \mapsto H^0(U_r,\mathbb{Z})$ on X. Recall that $\mathcal{I}^n$ denotes the Zariski sheaf on X associated to the presheaf $U \mapsto I^n(U)$. For any integer $n \geq 0$, the restriction of the global signature to the powers of the fundamental ideal of Definition (8.3) induces a homomorphism

$$\mathcal{I}^n \to \mathrm{supp}_* 2^n\mathbb{Z}$$

of Zariski sheaves on X. Similarly, $I^n(X) \overset{\langle\langle -1\rangle\rangle}{\to} I^{n+1}(X)$ induces a homomorphism $\mathcal{I}^n \overset{\langle\langle -1\rangle\rangle}{\to} \mathcal{I}^{n+1}$ of sheaves for any $n \geq 0$, and a homomorphism of sheaves

$$\varinjlim I^n \to \mathrm{supp}_* \mathbb{Z}$$

where $\varinjlim I^n$ denotes the direct limit of the sequence of sheaves $\mathcal{W} \overset{\langle\langle -1\rangle\rangle}{\to} \mathcal{I} \overset{\langle\langle -1\rangle\rangle}{\to} \mathcal{I}^2 \overset{\langle\langle -1\rangle\rangle}{\to} \cdots$ Similarly, the signature induces a morphism of sheaves

$$\mathcal{W}[\frac{1}{2}] \to \operatorname{supp}_*\mathbb{Z}[\frac{1}{2}]$$

where $\mathcal{W}[\frac{1}{2}]$ is the sheaf associated to the presheaf $U \mapsto W(U)[\frac{1}{2}]$ and $\operatorname{supp}_*\mathbb{Z}[\frac{1}{2}]$ is the sheaf $U \mapsto H^0(U_r, \mathbb{Z}[\frac{1}{2}])$

Theorem 8.5. *Let X be a scheme with 2 invertible in its global sections.*

(1) *The signature morphism of sheaves of Definition (8.4)*

$$\varinjlim \mathcal{I}^n \to \operatorname{supp}_* \mathbb{Z} \tag{12}$$

is an isomorphism.

(2) *The signature morphism of sheaves of Definition (8.4)*

$$\mathcal{W}[\frac{1}{2}] \to \operatorname{supp}_*\mathbb{Z}[\frac{1}{2}] \tag{13}$$

is an isomorphism.

(3) *The signature induces an isomorphism of short exact sequence of sheaves on X*

$$\begin{tikzcd} 0 \arrow[r] & \varinjlim \mathcal{I}^n \arrow[r] \arrow[d] & \mathcal{W}[\frac{1}{2}] \arrow[r] \arrow[d] & \varinjlim \mathcal{W}/\mathcal{I}^n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \operatorname{supp}_*\mathbb{Z} \arrow[r] & \operatorname{supp}_*\mathbb{Z}[\frac{1}{2}] \arrow[r] & \operatorname{supp}_*(\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

where $\mathcal{W}/\mathcal{I}^n$ denotes the sheaf associated to the presheaf $U \mapsto \mathcal{W}(U)/\mathcal{I}^n(U)$.

(4) *The signature induces an isomorphism of short exact sequence of sheaves on X*

$$\begin{tikzcd} 0 \arrow[r] & \varinjlim \mathcal{I}^n \arrow[r, "\langle\langle -1\rangle\rangle"] \arrow[d] & \varinjlim \mathcal{I}^n \arrow[r] \arrow[d] & \varinjlim \overline{\mathcal{I}^n} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \operatorname{supp}_*\mathbb{Z} \arrow[r, "2"] & \operatorname{supp}_*\mathbb{Z} \arrow[r] & \operatorname{supp}_*\mathbb{Z}/2 \arrow[r] & 0 \end{tikzcd}$$

where $\overline{\mathcal{I}^n}$ denotes the sheaf associated to the presheaf $U \mapsto \mathcal{I}^n(U)/\mathcal{I}^{n+1}(U)$.

Proof. Statements (1) and (2) follow immediately from the local case, Proposition (7.1) and Corollary (7.1) respectively, as it is sufficient to prove that they induce an isomorphism on stalks. As supp_* is exact, statements (3) and (4) may be obtained by applying supp_* to the analogous short exact sequences of groups and then using the two out of three lemma to conclude, but for (4), one should note that

$$\varinjlim \mathcal{I}^n \overset{\langle\langle -1\rangle\rangle}{\to} \varinjlim_{n\geq 1} \mathcal{I}^n$$

is an isomorphism in order to obtain exactness of the top row of the diagram in (4). □

The next corollary is an immediate consequence of the previous theorem and Lemma (4.1).

Corollary 8.1. *Let X be a scheme with 2 invertible.*

(1) *For any $m \geq 0$, the morphism (12) induces an isomorphism of cohomology groups*
$$H^m_{Zar}(X, \varinjlim \mathcal{I}^n) \to H^m(X_r, \mathbb{Z})$$
(2) *For any $m \geq 0$, the morphism (13) induces an isomorphism of cohomology groups*
$$H^m_{Zar}(X, \mathcal{W}[\frac{1}{2}]) \to H^m(X_r, \mathbb{Z}[\frac{1}{2}])$$

Corollary 8.2. *Let X be a scheme with 2 invertible which is quasi-separated and quasi-compact. Then, there is an isomorphism of cohomology groups for all $m \geq 0$*
$$\bigoplus_{m\geq 0} H^m_{Zar}(X, \varinjlim \overline{\mathcal{I}^n}) \simeq \varinjlim H^n_{\acute{e}t}(X, \mathbb{Z}/2)$$

Proof. Under the hypotheses stated C. Scheider has proved [Sch94, Corollary (7.19)] that there is an isomorphism
$$\varinjlim H^n_{\acute{e}t}(X, \mathbb{Z}/2) \overset{\simeq}{\to} \bigoplus_{m\geq 0} H^m(X_r, \mathbb{Z}/2)$$
and from Theorem 8.5 one has an isomorphism $H^m_{Zar}(X, \overline{\mathcal{I}^n}) \overset{\simeq}{\to} H^m(X_r, \mathbb{Z}/2)$ for all $m \geq 0$. Thus one obtains the isomorphism stated. □

Corollary 8.3. *If X is a real variety, by which we mean a scheme which is separated and of finite type over $\mathbb{R}$, and the Krull dimension of X is d, then whenever $n \geq d+1$ the signature induces an isomorphism in cohomology*
$$H^m_{Zar}(X, \mathcal{I}^n) \overset{\text{sign}}{\simeq} H^m_{sing}(X(\mathbb{R}), \mathbb{Z})$$
for all integers $m \geq 0$ and an isomorphism of long exact sequences as stated in the introduction.

Proof. It suffices to see that the morphism of sheaves $\mathcal{I}^n \overset{\langle\langle -1\rangle\rangle}{\to} \mathcal{I}^{n+1}$ is an isomorphism for $n \geq d+1$, for then multiplication by 2^{d+1} defines an isomorphism of sheaves $\mathcal{I}^{d+1} \simeq \varinjlim \mathcal{I}^n$ and hence we obtain the statement of the corollary using Theorem 8.5 in view of Remark (4.3). When $n \geq d+1$, for any U open in X we have an isomorphism of kernels in the diagram of residues below
$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I}^n(U) & \longrightarrow & I^n(K) & \longrightarrow & \bigoplus_{x\in X^{(1)}} I^{n-1}(k(x)) \\ & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} \\ 0 & \longrightarrow & \mathcal{I}^{n+1}(U) & \longrightarrow & I^{n+1}(K) & \longrightarrow & \bigoplus_{x\in X^{(1)}} I^n(k(x)) \end{array}$$
since the two rightmost vertical maps are isomorphisms for $n \geq d+1$, which proves the desired isomorphism. □

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