FINITENESS THEOREMS FOR THE SHIFTED WITT AND HIGHER GROTHENDIECK-WITT GROUPS OF ARITHMETIC SCHEMES

JEREMY JACOBSON

ABSTRACT. For smooth varieties over finite fields, we prove that the shifted (*aka* derived) Witt groups of surfaces are finite and the higher Grothendieck-Witt groups (*aka* Hermitian K-theory) of curves are finitely generated. For more general arithmetic schemes, we give conditional results, for example, finite generation of the motivic cohomology groups implies finite generation of the Grothendieck-Witt groups.

INTRODUCTION

The Witt group $W(X)$ of a scheme X was introduced by Knebusch [Kne77, Chapter 1 §5] in the seventies. When k is a field having characteristic different from 2, $W(\operatorname{Spec}(k))$ is the classical Witt group of quadratic forms over k. For varieties over finite fields, little is known in general about the Witt group, except that it is a torsion group (for example, see Corollary 3.3). One important result states that, when X is a complete regular curve over a finite field of characteristic different from 2, the Witt group is a finite group [AEJ94, Theorem 3.6].

More recently, the Witt group was revealed to be a part of a cohomology theory $W^n(X)$ for schemes. When 2 is invertible on X, each *shifted Witt group $W^n(X)$* can be constructed as the "triangular" Witt group [Bal00, Bal01a] of the triangulated category $D^b(\operatorname{Vect}(X))$ equipped with the shifted duality $Hom(-, \mathcal{O}_X[n])$, where $D^b(\operatorname{Vect}(X))$ denotes the bounded derived category of vector bundles $\operatorname{Vect}(X)$ on X. Recall the 4-periodicity, $W^n(X) \simeq W^{n+4}(X)$, and that they recover the classical Witt group as $W(X) \simeq W^0(X)$.

The first motivation for writing this article was to study the question of finiteness of these W^n for varieties over finite fields. We prove that when X is a smooth surface over a finite field of characteristic different from 2, the shifted Witt groups $W^n(X)$ are finite (see Theorem 3.10). In higher dimensions, we give conditional results. Theorem 3.11 states that, for X a finite type $\mathbb{Z}[\frac{1}{2}]$-scheme with no residue field of X formally real, if the motivic cohomology groups of X with mod 2 coefficients $H_{mot}^m(X, \mathbb{Z}/2\mathbb{Z}(n))$ are finite groups, then the shifted Witt groups $W^n(X)$ are finite. Furthermore, we give partial converses to this last result. We prove that for certain arithmetic schemes of dimension less than four, finiteness of the shifted Witt groups is equivalent to finiteness of the mod 2 motivic cohomology groups $H_{mot}^m(X, \mathbb{Z}/2\mathbb{Z}(n))$ (see Theorem 3.13).

The argument that we use for these results is essentially that of [AEJ94], but significantly strengthened by the fact that we now can use Voevodsky's solution of the Bloch-Kato conjecture. Indeed, let X be a smooth variety over a field of characteristic different from 2. Using Bloch-Kato, Gille noted that his graded Gersten-Witt spectral sequence relates étale cohomology to the Witt groups [Gil07, §10.7]. When

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the base field is $\mathbb{C}$, Totaro also used this spectral sequence, noting that it easily gave Parimala's theorem, equating finiteness of $CH^2(X)/2CH^2(X)$ to finiteness of $W^0(X)$ [Tot03, Theorem 1.4]. Here, we adapt these ideas to the arithmetic setting (smooth schemes over $\mathbb{Z}[\frac{1}{2}]$) using Arason's Theorem (Theorem 2.5). Also, we apply the positive solution of the Kato conjecture to relate finiteness of motivic cohomology with mod 2 coefficients to finiteness of the Witt groups for varieties having dimension as high as 4 (see Proposition 1.15 and Lemma 1.21 which lead to Theorem 3.13).

The second motivation relates to Schlichting's Grothendieck-Witt groups of schemes [Sch10a]. They form a bigraded cohomology theory $GW_m^n(X)$ for schemes which generalizes Knebusch's Grothendieck-Witt group $L(X)$ of a scheme X [Kne77, Chapter 1 §4] with $L(X) \simeq GW_0^0(X)$ [Sch10a, Proposition 4.11]. They are the algebraic analogue of real topological K-theory in the same way that algebraic K-theory is the algebraic analogue of complex topological K-theory. A major goal is to understand the Grothendieck-Witt groups of schemes at the same level as the higher algebraic K-groups $K_m(X)$.

Recall that the Bass conjecture states that the higher algebraic K-groups $K_m(X)$ of a regular finite type $\mathbb{Z}$-scheme X are finitely generated as abelian groups [Kah05, §4.7.1 Conjecture 36]. There are two main results on this conjecture:

(a) When $\dim(X) \leq 1$, Quillen proved the conjecture [Kah05, §4.7.1 Proposition 38 (b)];

(b) The "motivic" Bass conjecture, that is, finite generation of the motivic cohomology groups $H_{mot}^m(X, \mathbb{Z}(n))$ [Kah05, See §4.7.1 Conjecture 37], implies the Bass conjecture. This follows from the Atiyah-Hirzebruch spectral sequence [Kah05, §4.3.2 Equation (4.6) and the final paragraph of §4.6].

The second motivation for this article was to attempt to reproduce for the Grothendieck-Witt groups the two results above about K-theory. Regarding the Hermitian analogue of (a), finite generation of the Grothendieck-Witt groups was known to follow (e.g. Karoubi induction [BK05, Proposition 3.5]) from finiteness of the shifted Witt groups and finite generation of the higher algebraic K-groups. So, one immediate corollary of the finiteness result for Witt groups is a finite generation result for the Grothendieck-Witt groups of curves over finite fields. For the analogue of (b), up to the condition that we must assume that no residue field of X is formally real, we are successful. These results appear in Section 4.

Finally, a finiteness result for the Chow-Witt groups appears in Section 5. It has been observed (e.g. [Hor08], [FS09]) that the Chow-Witt groups appear on the second page of the coniveau spectral sequence for the p-th shifted Grothendieck-Witt groups as $E_2^{p,-p} \cong \widetilde{CH}^p(X)$. For the usual Chow groups, they appear in a similar way in the coniveau spectral sequence converging to K-theory, and there is a classical finiteness result stating that the d-th Chow group $CH^d(X)$ of a quasi-projective variety of dimension d over a finite field is finite [KS10, Corollary 9.4 (1)]. The result given here is the Chow-Witt analogue, stating that $\widetilde{CH}^d(X)$ is finite (Theorem 5.3).

Contents

1. Kato complexes, Kato cohomology, and motivic cohomology

For a very general class of schemes Kato [Kat86a, §1] introduced complexes that generalized to higher dimensions classical exact sequences for Galois cohomology. He made some conjectures on their exactness in various situations. In this section, we recall some finiteness results about their cohomology that are easily obtained using finiteness of étale cohomology, and we explain their relation to motivic cohomology.

1.1. **Kato complexes.** First a remark about the implications of assuming 2 is invertible.

Remark 1.1. Recall that when X is a scheme, we say that 2 *is invertible on X* when 2 is a unit in the global sections $\Gamma(X, \mathcal{O}_X)$.

  (a) When X is of finite type over $\mathbb{Z}$, saying that 2 is invertible on X is the same as saying that X is of finite type over $\mathbb{Z}[\frac{1}{2}]$. Furthermore, when X is of finite type over $\mathbb{F}_p$ $(p > 2)$, it follows that 2 is invertible on X and that X is of finite type over $\mathbb{Z}$ (as $\mathbb{F}_p$ is of finite type over $\mathbb{Z}$, and compositions of finite type morphisms are of finite type), hence X is of finite type over $\mathbb{Z}[\frac{1}{2}]$.
  (b) From the assumption 2 is invertible on X, it also follows that each residue field $k(x)$ of X has characteristic different from 2, and that there is an isomorphism of $\mathrm{Gal}(k(x)_s|k(x))$-modules, $\mathbb{Z}/2\mathbb{Z} \simeq \mu_2 := \{a \in k(x)_s | a^2 = 1\}$, where $k(x)_s$ denotes a separable closure of $k(x)$.
  (c) When 2 is invertible on X, in the global sections $-1 \neq 1$, so -1 determines an isomorphism of étale sheaves $\mu_2 \simeq \mathbb{Z}/2\mathbb{Z}$, hence $\mu_2^{\otimes n}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Recall that on a scheme X, isomorphisms between the étale sheaf μ_2 and the constant sheaf $\mathbb{Z}/2\mathbb{Z}$ correspond to global sections of X which have order 2 on each connected component [Tam94, see p.100 for definitions and details].

Next, we recall what we mean by the residue and corestriction maps.

Definition 1.2. Let A be a discrete valuation ring (DVR) with fraction field K and residue field k. Assume that $\mathrm{char}(k) \neq 2$. By *the residue homomorphism for A*, we mean the group homomorphism from the Galois cohomology of K to the Galois cohomology of the residue field k

$$\partial_n : H^n_{Gal}(K, \mathbb{Z}/2\mathbb{Z}) \to H^{n-1}_{Gal}(k, \mathbb{Z}/2\mathbb{Z}),$$

as defined in [Kat86a, p.149]. Note that this is the same as the definition given in [Ara75, p.475]. When $n = 0$, we set $H^{n-1}(k, \mathbb{Z}/2\mathbb{Z}) = 0$.

Definition 1.3. Let F be a finite extension of a field L. By *the corestriction homomorphism for the finite extension* L/K, we mean the group homomorphism

$$cor_{L/K} : H^n_{Gal}(L, \mathbb{Z}/2\mathbb{Z}) \to H^n_{Gal}(K, \mathbb{Z}/2\mathbb{Z}),$$

as defined in [Ara75, p.471]. This agrees with the definition used by Kato in [Kat86a].

Recall that a locally noetherian scheme X is said to be excellent [Gro65, Definition 7.8.5]) if for some covering of X by affine schemes $U_\alpha =$ Spec(A_α), each of the rings A_α are excellent [Gro65, Definition 7.8.2].

The next example is important for understanding the definition of the differentials in the Kato complex.

Example 1.4. Let X be a noetherian excellent scheme, let $y \in X$ be a point of X, and let $Z := \overline{\{y\}}$ denote the reduced closed subscheme with underlying topological space $\overline{\{y\}}$. Since X is excellent, every locally finite-type X-scheme X' is excellent [Gro65, Proposition 7.8.6]. In particular, the closed immersion $Z \to X$ is excellent. Therefore, Z is an integral excellent scheme. For an integral excellent ring A, its integral closure in Frac(A) is a finite A-algebra [Gro65, Scholie 7.8.3]. It follows that the normalization morphism $Z' \to Z$ is finite. In particular, the normalization is quasi-finite, so the fiber over any point $x \in Z$ has only finitely many points $x_1, \ldots, x_n$ and for each of the x_i the field extension $\kappa(x_i)/\kappa(x)$ is a finite extension.

Definition 1.5. Let X be a scheme. Recall that the *dimension of a point* $x \in X$ is defined to be the (combinatorial) dimension $\dim(x) := \dim(\overline{\{x\}})$ of the topological space defined by the closure of x. The set of dimension p points of X is denoted by X_p. The *codimension of a point* $x \in X$ is defined to be the Krull dimension $\text{codim}(x) := \dim(\mathcal{O}_{X,x})$ of the local ring of X at the point $x \in X$. This is equal to the topological codimension of the closed subspace $\overline{\{x\}}$ in X [Gro65, Proposition 5.12]. The set of codimension p points of X will be denoted by X^p.

Definition 1.6 (The yx-component ∂_{yx} of the differential)**.** Let X be a noetherian excellent scheme with 2 invertible. Recall the facts of Example 1.4. Let $x \in X^{p+1}$ and $y \in X^p$ such that $x \in \overline{\{y\}}$. Let $Z := \overline{\{y\}}$ denote the reduced closed subscheme with underlying topological space $\overline{\{y\}}$. Let $Z' \to Z$ be the normalization of Z. The field extensions $\kappa(x_i)/\kappa(x)$ for each of the finitely many points $x_1, \ldots, x_n \in Z'$ lying over $x \in Z$ are finite extensions, so for all non-negative integers $j \in \mathbb{Z}$, there are well-defined corestriction maps (Definition 1.3) $cor_{\kappa(x_i)/\kappa(x)} : H^j(\kappa(x_i), \mathbb{Z}/2\mathbb{Z}) \to H^j(\kappa(x), \mathbb{Z}/2\mathbb{Z})$. Each $x_i \in Z'$ is of codimension 1 in Z' (use the dimension formula [Mat89, Theorem 15.6] together with the fact that the extension $\kappa(x_i)/\kappa(x)$ is finite, hence of transcendence degree 0). So the localization $\mathcal{O}_{Z',x_i}$ of the normalization at the point x_i is a DVR, with fraction field $\kappa(y)$ and residue field $\kappa(x_i)$. Hence, each x_i defines residue homomorphisms (Definition 1.2)

$$\partial^{x_i} : H^j_{Gal}(\kappa(y), \mathbb{Z}/2\mathbb{Z}) \to H^{j-1}_{Gal}(\kappa(x_i), \mathbb{Z}/2\mathbb{Z})$$

for all non-negative integers $j \in \mathbb{Z}$. The yx-component d_{yx} is defined (c.f. [Jan, §0.6]) as

$$d_{yx} := \sum_{x_i | x} cor_{\kappa(x_i)/\kappa(x)} \circ \partial^{x_i},$$

where the sum is taken over the finitely many points $x_i \in Z'$ lying over x.

Definition 1.7 (cohomological Kato complexes)**.** Let X be a noetherian excellent scheme, finite dimensional of dimension d. We assume that 2 is invertible on X (this is not necessary for the definition in general). It follows from this assumption that for every $n \geq 0$ the Tate twist $\mu_2^{\otimes n}$ is isomorphic to the constant sheaf $\mathbb{Z}/2\mathbb{Z}$ (see Remark 1.1 (b)). The n-th Kato complex is defined to be the complex

$$C(X, H^n) := \bigoplus_{x \in X^0} H^n(\kappa(x), \mathbb{Z}/2\mathbb{Z}) \to \bigoplus_{x \in X^1} H^{n-1}(\kappa(x), \mathbb{Z}/2\mathbb{Z}) \to \cdots$$

$$\cdots \to \bigoplus_{x \in X^d} H^{n-d}(\kappa(x), \mathbb{Z}/2\mathbb{Z}),$$

where $\kappa(x)$ denotes the residue field of a point $x \in X$, and we set $H^m(\kappa(x), \mathbb{Z}/2\mathbb{Z}) = 0$ for $m < 0$. Kato complexes are often indexed homologically, but here we will always use cohomological indexing by placing the term summing over the codimension p points in degree p. The m-th cohomology of the n-th Kato complex $C(X, H^n)$ will be denoted by $H^m(C(X, H^n))$. The differential is defined componentwise. The yx-component $\partial_i^{yx} : H^i(\kappa(y), \mathbb{Z}/2\mathbb{Z}) \to H^{i-1}(\kappa(x), \mathbb{Z}/2\mathbb{Z})$ of the i-th differential is defined as follows: If $x \notin \overline{\{y\}}$, then set $d_{yx} = 0$. If $x \in \overline{\{y\}}$, then d_{yx} is defined as in Definition 1.6.

Remark 1.8. When X is a variety over a field, by definition the Kato complexes are the same as Rost's cycle complexes for the cycle module defined by Galois cohomology [Ros96, see §2.10 for the definition of the differential, as well as Remark (2.5)].

Finally, we recall some conditions under which dimension can be replaced by codimension in the definition of the Kato complexes.

Definition 1.9. Let X be a scheme. The scheme X is said to be *biequidimensional* [Gro64, §14 p.11] if it is finite dimensional, equidimensional (*aka* pure, i.e. the dimension of each irreducible component is the same), equicodimensional (the codimension of each minimal closed irreducible set in X is the same), and catenary (see [Gro64, §14 p.11]).

Lemma 1.10 (Corollaire 14.3.5 EGA IV Premiére Partie)**.** *For any noetherian biequidimensional scheme X of dimension d and for any point $x \in X$, the dimension and codimension of x are related as follows:* $\dim(x) = d - \text{codim}(x)$. *That is for any p, the set of dimension p points of X is equal to the set of codimension $d - p$ points $X_p = X^{d-p}$.*

Remark 1.11. There are examples of schemes which are regular (hence catenary) and integral (hence equidimensional) possessing points x for which $\dim(x)+\text{codim}(x)$ is not equal to the dimension of the scheme [Gro65, Remark 5.2.5 (i)]. However, when X is a variety over a field, pure of dimension d, it is biequidimensional [Gro65, follows from Proposition 5.2.1].

1.2. **Relation to étale cohomology.** Jannsen, Saito, and Sato showed that for very general schemes, the Kato complexes appear on the first page of the étale niveau spectral sequence. As we restate their result slightly to suit our purposes, we recall briefly their proof.

Proposition 1.12. [Jan, Section 1.5 and Theorem 1.5.3] *Let X be a noetherian regular excellent $\mathbb{Z}[\frac{1}{2}]$-scheme, pure of dimension d. Filtering by codimension of support gives a convergent spectral sequence*

$$E_1^{p,q}(X, \mathbb{Z}/2\mathbb{Z}) := \bigoplus_{x \in X^p} H_{\acute{e}t}^{q-p}(\kappa(x), \mathbb{Z}/2\mathbb{Z}) \Longrightarrow H_{\acute{e}t}^{p+q}(X, \mathbb{Z}/2\mathbb{Z}).$$

with differential d_r of bidegree $(r, 1-r)$. Furthermore, the complexes appearing on the first page of the spectral sequence $E^1_{*,q}(X, \mathbb{Z}/2\mathbb{Z})$ agree up to signs with the Kato complexes $C(X, H^q)$, hence the second page of the spectral sequence consists of Kato cohomology $E^{p,q}_2 = H^p(C(X, H^q))$.

Proof. For any noetherian scheme, pure of dimension d, one may construct a cohomological spectral sequence of the form (e.g. , [CTHK97, Section 1])

$$E^{p,q}_1(X, \mathbb{Z}/2\mathbb{Z}) := \bigoplus_{x \in X^p} H^{p+q}_x(X, \mathbb{Z}/2\mathbb{Z}) \Longrightarrow H^{p+q}_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}).$$

where $H^{p+q}_x(X, \mathbb{Z}/2\mathbb{Z})$ is defined to be the colimit, over all non-empty open subsets $U \subset X$ containing x, of the groups $H^{p+q}_{\overline{\{x\}} \cap U}(U, \mathbb{Z}/2\mathbb{Z})$. Since X, and hence $\overline{\{x\}}$, is excellent, there exists an open $U_0 \subset X$ such that $\overline{\{x\}} \cap U$ is regular for $U \subset U_0$. In this situation, if $x \in X^p$, then $\overline{\{x\}} \cap U$ is a codimension p embedding in U, hence by absolute purity [Fuj02, Theorem 2.1]

$$H^{p+q-2p}(\overline{\{x\}} \cap U, \mathbb{Z}/2\mathbb{Z}) \simeq H^{p+q}_{\overline{\{x\}} \cap U}(U, \mathbb{Z}/2\mathbb{Z})$$

so

$$E^{p,q}_1(X, \mathbb{Z}/2\mathbb{Z}) \simeq \bigoplus_{x \in X^p} H^{q-p}_{\text{ét}}(\kappa(x), \mathbb{Z}/2\mathbb{Z})$$

For any $y \in X^p$, $x \in X^{p+1}$, the yx-components of the differentials

$$H^{q-p}_{\text{ét}}(\kappa(y), \mathbb{Z}/2\mathbb{Z}) \to H^{q-p-1}_{\text{ét}}(\kappa(x), \mathbb{Z}/2\mathbb{Z})$$

commute, up to signs, with those of the Kato complex [Jan, Theorem 1.1.1]. This completes the proof. $\square$

Corollary 1.13. *Let X be a separated scheme that is smooth (i.e. formally smooth and of finite type) over $\mathbb{Z}[\frac{1}{2}]$, pure of dimension d. The spectral sequence of the proposition above takes the form*

$$E^{p,q}_2(X, \mathbb{Z}/2\mathbb{Z}) := H^p_{Zar}(X, \mathcal{H}^q) \Longrightarrow H^{p+q}_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}),$$

where $H^p_{Zar}(X, \mathcal{H}^q)$ denotes the Zariski cohomology of the Zariski sheaf $\mathcal{H}^q$ associated to the presheaf $U \mapsto H^q_{\text{ét}}(U, \mathbb{Z}/2\mathbb{Z})$. Hence, for all $p, q \in \mathbb{Z}$ the Kato cohomology groups $H^p(C(X, H^q))$ and the Zariski cohomology groups $H^p_{Zar}(X, \mathcal{H}^q)$ agree.

Proof. To prove the corollary, it suffices to show that the sheaf of complexes associated to the presheaves $U \mapsto E^{*,q}_1(U, \mathbb{Z}/2\mathbb{Z})$ is a flasque resolution of $\mathcal{H}^q$. A complex of sheaves is exact if and only if it is exact on stalks. So, it suffices to demonstrate that, for every point $x \in X$, the complex $E^{*,q}_1(\mathcal{O}_{X,x}, \mathbb{Z}/2\mathbb{Z})$ is exact in positive degrees and in degree zero $E^{0,q}_2 \simeq H^q_{\text{ét}}(\mathcal{O}_{X,x}, \mathbb{Z}/2\mathbb{Z})$. This is known as the Gersten conjecture. Since the morphism $X \to \text{Spec}(\mathbb{Z}[\frac{1}{2}])$ is smooth, the local ring $\mathcal{O}_{X,x}$ is formally smooth and essentially of finite type over $\mathcal{O}_{\mathbb{Z}[\frac{1}{2}],y}$. The ring $\mathcal{O}_{\mathbb{Z}[\frac{1}{2}],y}$ is either a DVR or a field. In both cases, the Gersten conjecture is known. For the field case see, for example [BO74], and in the DVR case it was proved by Gillet [Gil]. $\square$

Lemma 1.14. *Let X be a smooth variety (i.e. separated, formally smooth and of finite type) over a finite field $\mathbb{F}_p$ ($p > 2$), pure of dimension d. For any codimension p point $x \in X^p$ of X, we have $\text{cd}_2(\kappa(x)) \leq 1 + d - p$, where $\text{cd}_2(\kappa(x))$ denotes the étale cohomological 2-dimension of the residue field $\kappa(x)$ of x. Considering the E_1 entries of the coniveau spectral sequence, if $q > d + 1$, then $E^{p,q}_1 = 0$ for all $p \in \mathbb{Z}$, and hence the Kato complex $C(X, H^q)$ vanishes.*

Proof. Let $x \in X^p$ be a codimension p point of X. By [SGA73, Exposé X, Theorem 2.1] $\mathrm{cd}_2(\kappa(x)) \leq 1 + \mathrm{tr.deg}_{\mathbb{F}_p} \kappa(x)$, where $\mathrm{cd}_2(\kappa(x))$ denotes the étale cohomological 2-dimension of the residue field $\kappa(x)$ of x. As X is of finite type over a field, one has [Gro65, Corollaire 5.2.3] that $\dim_x(X) = \dim(\mathcal{O}_{X,x}) + \mathrm{tr.deg}_{\mathbb{F}_p} \kappa(x)$. It results from [Gro64, Proposition 14.1.6] that $d = \dim(X) \geq \dim_x(X)$ for all $x \in X$. Hence, $d - p \geq \mathrm{tr.deg}_{\mathbb{F}_p} \kappa(x)$, proving that $\mathrm{cd}_2(\kappa(x)) \leq 1 + d - p$. This proves the lemma. $\square$

1.3. Finiteness results for Kato cohomology. The Kato conjecture was recently solved. We write it down for later reference.

Proposition 1.15. [KS10, Theorem 8.1] *Let X be a regular connected scheme of Krull dimension d, proper over a finite field $\mathbb{F}_p$ ($p > 2$). The Kato cohomology groups $H^p(C(X, H^{d+1}))$ vanish except when $p = d$, in which case $H^d(C(X, H^{d+1})) \simeq \mathbb{Z}/2\mathbb{Z}$.*

Next, recall the following fact about étale cohomology.

Lemma 1.16. *Let X be a regular separated scheme of finite type over $\mathbb{Z}[\frac{1}{2}]$. In this situation, the étale cohomology groups $H^m_{\acute{e}t}(X, \mathbb{Z}/2\mathbb{Z})$ are finite groups for all $m \geq 0$.*

Proof. Let f denote the structural morphism $X \to \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$. From the finiteness theorem [Del77, Théorèmes de Finitude, §1, Theorem 1.1] we have that, for all $q \geq 0$, the étale sheaves $R^q f_* \mathbb{Z}/2\mathbb{Z}$ are constructible. Using the Leray spectral sequence $E_1^{p,q} = H^p_{\acute{e}t}(\mathbb{Z}[\frac{1}{2}], R^q f_* \mathbb{Z}/2\mathbb{Z}) \Rightarrow H^{p+q}_{\acute{e}t}(X, \mathbb{Z}/2\mathbb{Z})$ [Del77, Cohomologie étale, §2, p.6], we reduce to proving that the étale cohomology groups of $\mathbb{Z}[\frac{1}{2}]$ with coefficients in a constructible sheaf are finite, which is known [Mil86, Chapter 2 §3 Theorem 3.1 and following discussion] $\square$

Finally, we recall the following well known finiteness results for Kato cohomology, which use the finiteness of étale cohomology together with the coniveau spectral sequence.

Lemma 1.17 (Absolute Finiteness). *Let X be a pure regular separated scheme of finite type over a base scheme S. Consider the following situations:*

(a) *$dim(X) \leq 1$, and $S = \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$;*
(b) *$dim(X) \leq 2$, X is quasi-projective over S, and $S = \mathrm{Spec}(\mathbb{F}_p)$ ($p > 2$).*
(c) *$dim(X) = d$, and $S = \mathrm{Spec}(\mathbb{F}_p)$ ($p > 2$).*
(d) *$dim(X) = d$, X is quasi-projective, and $S = \mathrm{Spec}(\mathbb{F}_p)$ ($p > 2$).*

In situations (a) and (b), all the Kato cohomology groups of X are finite. In situation (c), the Kato cohomology group $H^d(C(X, H^{d+1}))$ is finite, and in situation (d), the Kato cohomology group $H^d(C(X, H^d))$ is finite

Proof. In all situations, X satisfies the hypotheses of Lemma 1.16, hence the étale cohomology of X is finite. Now consider the coniveau spectral sequence for étale cohomology of Proposition 1.12. In situation (a), all differentials on the second page of the spectral sequence are zero because $\dim(X) \leq 1$. So, the spectral sequence collapses on the second page. Hence, as the abutment is finite (Lemma 1.16), the Kato cohomology groups are finite.

For (b), using Lemma 1.14 we see that there is only one possibly non-zero differential $d_2 : H^0(C(X, H^3)) \to H^2(C(X, H^2))$ on the second page of the spectral sequence. It follows that all the other Kato cohomology groups appear on the stable page of the spectral sequence, hence are quotients of the induced filtration on étale cohomology, so they are finite. The kernel and cokernel of d_2 appear on the stable page, hence are finite. Therefore $H^0(C(X, H^3))$ is finite if and only if $H^2(C(X, H^2))$ is finite. The group $H^2(C(X, H^2))$, that is, $E_2^{2,2}$, is isomorphic to

the mod 2 Chow group $CH^d(X)/2$ of codimension d cycles [BO74, Theorem 7.7], which is finite for X quasi-projective over a finite field [KS10, Corollary 9.4 (1)].

Now, assume we are in situation (c). From Lemma 1.14, it follows that the differentials on the second page of the spectral sequence that are entering and leaving the group $H^d(C(X, H^{d+1}))$ are zero. So, $H^d(C(X, H^{d+1}))$ appears on the stable page, hence is finite since the abutment is finite.

Finally, assume we are in situation (d). As in the proof of case (b), the group $E_2^{d,d} = H^d(C(X, H^d))$ is isomorphic to $CH^d(X)/2$ [BO74, Theorem 7.7], hence is finite for X quasi-projective over a finite field [KS10, Corollary 9.4 (1)]. This completes the proof of the lemma. □

1.4. **Relation to motivic cohomology.** First recall the definition of the motivic cohomology of a smooth scheme over a Dedekind domain. Let X be a scheme that is separated and smooth over a Dedekind domain D. The standard algebraic m-simplex will be denoted by

$$\triangle_D^m := \mathrm{Spec}(D[t_0, t_1, \ldots, t_m]/\Sigma_i t^i - 1)$$

and the free abelian group on closed integral subschemes of codimension n in $X \times_D \triangle_D^m$, which intersect all faces properly, will be denoted by $z^n(X, m)$. Placing $z^n(X, 2n - m)$ in degree m, the associated complex of presheaves is denoted by $\mathbb{Z}(n)$, and we set $\mathbb{Z}/2\mathbb{Z}(n) := \mathbb{Z}(n) \otimes^{\mathbb{L}} \mathbb{Z}/2\mathbb{Z}$. The complex $\mathbb{Z}/2\mathbb{Z}(n)$ is in fact a complex of sheaves for the étale topology [Gei04, Lem. 3.1], and when considered as a complex of sheaves for the Zariski topology it will be denoted by $\mathbb{Z}/2\mathbb{Z}(n)_{Zar}$.

Definition 1.18. The *motivic cohomology groups of X with mod 2 coefficients* $H_{mot}^m(X, \mathbb{Z}/2Z(n))$ are defined to be the hypercohomology groups of the complex of Zariski sheaves $\mathbb{Z}/2Z(n)_{Zar}$.

Remark 1.19. In this remark we explain an observation of Totaro's [Tot03, Theorem 1.3 and surrounding discussion], that the Beilinson-Lichtenbaum conjecture leads to the long exact sequence 1.1 below. Let X be a separated scheme that is smooth over $D := \mathbb{Z}[\frac{1}{2}]$. Let $\pi : (Sm/D)_{\acute{e}t} \to (Sm/D)_{Zar}$ denote the natural morphism of sites.

(a) By the *Beilinson-Lichtenbaum conjecture with $\mathbb{Z}/2\mathbb{Z}$-coefficients*, we mean that there is a quasi-isomorphism $(\mathbb{Z}/2\mathbb{Z}(n))_{Zar} \simeq \tau_{\leq n} R\pi_* \mathbb{Z}/2\mathbb{Z}$ of complexes of Zariski sheaves on X. Recall that $R\pi_* \mathbb{Z}/2\mathbb{Z}$ is the complex of Zariski sheaves obtained by first taking an injective resolution $I^\bullet$ of the étale sheaf $\mathbb{Z}/2\mathbb{Z}$, from which we obtain an exact complex of étale sheaves. Then, applying π_* to this complex to obtain a complex of Zariski sheaves (no longer exact). The cohomology of this complex in degree i is the right derived functor $R^i \pi_* \mathbb{Z}/2\mathbb{Z}$, which is isomorphic to the Zariski sheaf $\mathcal{H}^i$ associated to the presheaf $U \mapsto H_{\acute{e}t}^i(U, \mathbb{Z}/2\mathbb{Z})$ [Tam94, I Proposition 3.7.1]. The complex $\tau_{\leq n} R\pi_* \mathbb{Z}/2\mathbb{Z}$ is a complex of Zariski sheaves with cohomology in degree i equal to $R^i \pi_* \mathbb{Z}/2\mathbb{Z}$ when $i \leq n$ and zero otherwise. It follows that there is a distinguished triangle in the derived category of Zariski sheaves on X

$$\tau_{\leq n-1} R\pi_* \mathbb{Z}/2\mathbb{Z} \to \tau_{\leq n} R\pi_* \mathbb{Z}/2\mathbb{Z} \to \mathcal{H}^n[-n]$$

then from the associated long exact sequence in hypercohomology, if the Beilinson-Lichtenbaum conjecture with $\mathbb{Z}/2\mathbb{Z}$-coefficients holds, we obtain the long exact sequence (c.f. [Tot03, Theorem 10.3])

(1.1) $\cdots \to H_{mot}^{m+n}(X, \mathbb{Z}/2\mathbb{Z}(n-1)) \to H_{mot}^{m+n}(X, \mathbb{Z}/2\mathbb{Z}(n)) \to H_{Zar}^m(X, \mathcal{H}^n) \to$
$H_{mot}^{m+n+1}(X, \mathbb{Z}/2\mathbb{Z}(n-1)) \to \cdots$

where $H_{Zar}^m(X, \mathcal{H}^n)$ denotes the Zariski cohomology of the Zariski sheaf $\mathcal{H}^n$ associated to the presheaf $U \mapsto H_{\acute{e}t}^n(U, \mathbb{Z}/2\mathbb{Z})$.

(b) For smooth schemes over fields, the Beilinson-Lichtenbaum conjecture is known, since it is equivalent to the Bloch-Kato conjecture [Kah05, Theorem 19], and the Bloch-Kato conjecture is known [Kah05, see Theorem 21 and surrounding discussion for an overview].

Lemma 1.20. *Let X be a pure separated scheme that is smooth over $\mathbb{Z}[\frac{1}{2}]$. Recall that $\mathcal{H}^q$ denotes the Zariski sheaf associated to the presheaf $U \mapsto H_{\acute{e}t}^q(U, \mathbb{Z}/2Z)$. Consider the following statements:*

(a) *The Zariski cohomology groups $H_{Zar}^p(X, \mathcal{H}^q)$ are finite for all $p, q \in \mathbb{Z}$;*

(b) *The motivic cohomology groups $H_{mot}^p(X, \mathbb{Z}/2\mathbb{Z}(q))$ are finite for all $p, q \in \mathbb{Z}$;*

(c) *The Beilinson-Lichtenbaum conjecture with $\mathbb{Z}/2\mathbb{Z}$-coefficients is true (see Remark 1.19).*

We have the implication (a) implies (b). Furthermore if we assume (c), then (a) is equivalent to (b). Hence, by Corollary 1.13, (b) is equivalent to finiteness of the Kato cohomology groups $H^p(C(X, H^q))$ for all $p, q \in \mathbb{Z}$.

Proof. To prove that (a) implies (b), recall that there is a coniveau spectral sequence [Gei04, see §4 for integral coefficients version]

$$E_1^{p,q}(X, \mathbb{Z}/2\mathbb{Z}(n)_{Zar}) := \bigoplus_{x \in X^p} H_{mot}^{2n-p+q}(\kappa(x), \mathbb{Z}/2\mathbb{Z}(n-p)) \implies H_{mot}^{p+q}(X, \mathbb{Z}/2\mathbb{Z}(n)).$$

and the sheaf of complexes associated to the presheaf $U \mapsto E_1^{*,q}(U, \mathbb{Z}/2\mathbb{Z}(n)_{Zar})$ gives a flasque resolution of the sheaf $\mathcal{H}^q$ [Gei04, Theorem 1.2 (2),(4) and (5), also see remark at start of page 775], hence the Zariski cohomology groups $H_{Zar}^p(X, \mathcal{H}^q)$ are the only groups on the E_2 page of the above spectral sequence, which converges to the motivic cohomology groups, and it follows that (a) implies (b).

Now assume (c), from which we obtain the long exact sequence 1.1 (see Remark 1.19), from which it follows that (b) implies (a). $\qquad\square$

Next we recall some finiteness theorems relating the Kato cohomology to motivic cohomology in the cases that finiteness of these groups is only partially known.

Lemma 1.21 (Relative Finiteness). *Let X be a pure separated scheme that is smooth over a base scheme S. Consider the following situations:*

(a) *$dim(X) \leq 2$, no residue field of X is formally real, $S = \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$;*

(b) *$dim(X) \leq 3$, X is connected and proper over S, and $S = \mathrm{Spec}(\mathbb{F}_p)$ $(p > 2)$;*

(c) *$dim(X) \leq 4$, X is connected and proper over S, and $S = \mathrm{Spec}(\mathbb{F}_p)$ $(p > 2)$;*

In situation (a):

(i) *The groups $H_{Zar}^0(X, \mathcal{H}^3)$, $H_{Zar}^0(X, \mathcal{H}^4)$, and $H_{Zar}^0(X, \mathcal{H}^5)$ are finite if and only if the groups $H_{Zar}^2(X, \mathcal{H}^2)$, $H_{Zar}^2(X, \mathcal{H}^3)$, and $H_{Zar}^2(X, \mathcal{H}^4)$ are finite. Furthermore, all the other groups $H_{Zar}^p(X, \mathcal{H}^q)$ are finite.*

(ii) *If all the groups $H_{Zar}^0(X, \mathcal{H}^3)$, $H_{Zar}^0(X, \mathcal{H}^4)$, and $H_{Zar}^0(X, \mathcal{H}^5)$ are finite, then the motivic cohomology groups $H_{mot}^p(X, \mathbb{Z}/2\mathbb{Z}(p))$ are finite for all $p, q \in \mathbb{Z}$. Assuming the Beilinson-Lichtenbaum conjecture (see Remark 1.19 for an explanation of what we mean by this), the converse is true.*

In situation (b):

(i) *The group $H_{Zar}^0(X, \mathcal{H}^3)$ is finite if and only if the group $H_{Zar}^2(X, \mathcal{H}^2)$ is finite. Furthermore, all the other cohomology groups $H_{Zar}^p(X, \mathcal{H}^q)$ are finite.*

(ii) The motivic cohomology groups $H^p_{mot}(X, \mathbb{Z}/2\mathbb{Z}(p))$ are finite for all $p, q \in \mathbb{Z}$ if and only if the group $H^0_{Zar}(X, \mathcal{H}^3)$ is finite.

In situation (c):

(i) The groups $H^2_{Zar}(X, \mathcal{H}^2)$, $H^2_{Zar}(X, \mathcal{H}^3)$, and $H^3_{Zar}(X, \mathcal{H}^3)$ are finite if and only if all the groups $H^0_{Zar}(X, \mathcal{H}^3)$, $H^0_{Zar}(X, \mathcal{H}^4)$, and $H^1_{Zar}(X, \mathcal{H}^4)$ are finite. Furthermore, all the other groups $H^p_{Zar}(X, \mathcal{H}^q)$ are finite.

(ii) The motivic cohomology groups $H^p_{mot}(X, \mathbb{Z}/2\mathbb{Z}(p))$ are finite for all $p, q \in \mathbb{Z}$ if and only if all the groups $H^0_{Zar}(X, \mathcal{H}^3)$, $H^0_{Zar}(X, \mathcal{H}^4)$, and $H^1_{Zar}(X, \mathcal{H}^4)$ are finite.

Proof. In all situations, for (ii), finiteness of motivic cohomology implies finiteness of the Zariski cohomology groups by Lemma 1.20. Also, in each situation, to prove (ii), it suffices to prove (i), for if the groups named in (i) are finite then all the groups $H^p_{Zar}(X, \mathcal{H}^q)$ are finite, hence the motivic cohomology groups are finite by Lemma 1.20. To prove (i), we work with the coniveau spectral sequence for étale cohomology of Proposition 1.12.

First assume that we are in situation (a). The étale cohomological 2-dimension of X is less than or equal to $2\dim(X) + 1$ [SGA73, Exposé 5, §6, Theorem 6.2], from which it follows that whenever $q > 2\dim(X) + 1$, we obtain vanishing of the Zariski sheaf $\mathcal{H}^q$, and hence vanishing of $H^p_{Zar}(X, \mathcal{H}^q)$. This, together with the fact that $\dim(X) \leq 2$, gives that there are only three possibly non-zero differentials on the second page of the spectral sequence, each having domain and codomain one of the groups named in (i). This proves the second statement of (i). To prove the first statement of (i), we prove finiteness of the kernel and cokernel of these differentials. To prove this claim, observe that the kernel and cokernel of these differentials appear on the third page of the spectral sequence, and the spectral sequence collapses on the third page. As the abutment is finite (Lemma 1.16) this proves the claim, finishing the proof of (i).

Assume that we are in situation (b). Then Lemma 1.14, Proposition 1.15, and the fact that $\dim(X) \leq 3$, give that there is only one possible non-zero differential on the second page of the spectral sequence. The domain of this differential is the group $H^0_{Zar}(X, \mathcal{H}^3)$. By the same argument used in the previous situation, this differential has finite kernel and cokernel, which concludes the proof in situation (b).

Finally, assume that we are in situation (c). Again, Lemma 1.14, Proposition 1.15, and the fact that $\dim(X) \leq 4$, give that there are only three possibly non-zero differentials on the second page of the spectral sequence, each having domain one of the groups named in the lemma. As before, the kernel and cokernel of these differentials are finite. Hence, this concludes the proof in the case of situation (c). □

2. Arason's theorem

In this section, for an excellent scheme X with 2 invertible, we recall the definition of the complex of abelian groups $C(X, \overline{I}^n)$. Arason essentially showed in [Ara75] that if the Bloch-Kato conjecture is true, then $C(X, \overline{I}^n)$ is isomorphic to the Kato complex $C(X, H^n)$. We call this result Arason's Theorem (see Theorem 2.5).

We first recall the definitions of the maps e^n, s^n, and h^n, relating Galois cohomology, Witt groups, and Milnor K-theory.

2.1. Galois cohomology: Definition of h^1.
Let k be a field having char $(k) \neq 2$ and let G denote the absolute Galois group $G := \mathrm{Gal}\,(k_s/k)$, where k_s denotes a separable closure of k. Let $\mu_2 := \left\{ a \in k_s | a^2 = 1 \right\}$ denote the group of square roots

of unity in k_s. Let $\mathbf{G}_m$ denote the multiplicative group k_s^* of units of k_s. The exact sequence of G-modules

$$1 \to \mu_2 \to \mathbf{G}_m \overset{2}{\to} \mathbf{G}_m \to 1,$$

where 2 denotes the endomorphism $x \mapsto x^2$, induces a long exact sequence in cohomology, from which we obtain the exact sequence (as G acts by evaluation on elements of $\mathbf{G}_m$, $H^0_{Gal}(k, \mathbf{G}_m) = k^*$)

$$(2.1) \qquad k^* \overset{2}{\to} k^* \to H^1(k, \mu_2) \to H^1_{Gal}(k, \mathbf{G}_m).$$

Since $H^1_{Gal}(k, \mathbf{G}_m) = 0$ [Ser94, Chapter 1 §1.2 Proposition 1], the exact sequence (2.1) induces the isomorphism

$$(2.2) \qquad k^*/k^{*2} \overset{\cong}{\to} H^1_{Gal}(k, \mu_2).$$

After identifying μ_2 with $\mathbb{Z}/2\mathbb{Z}$, the isomorphism (2.2) is denoted by

$$(2.3) \qquad h_k^1 : k^*/k^{*2} \overset{\cong}{\to} H^1_{Gal}(k, \mathbb{Z}/2\mathbb{Z}),$$

and is said to be the *norm-residue homomorphism in degree one*.

2.2. **Witt groups: Definition of s^1.** Let k be a field having char $(k) \neq 2$. The fundamental ideal $I(k)$ is defined to be the kernel of the mod 2 rank map $W(k) \to \mathbb{Z}/2\mathbb{Z}$. The q-th quotients $I^q(k)/I^{q+1}(k)$ of the powers of the fundamental ideal will be denoted by $\overline{I}^q(k)$. Taking the quotient by the kernel of the surjective dimension homomorphism induces an isomorphism $\overline{I}^0(k) = W(k)/I(k) \cong \mathbb{Z}/2\mathbb{Z}$.

Every unit $a \in k^*$ determines a non-degenerate symmetric bilinear form on k given by $\mathfrak{b}(k_1, k_2) := ak_1k_2$, and this form is denoted by $\langle a \rangle$, and the orthogonal sum of n such forms $\langle a_i \rangle$, where $a_i \in k^*$, is denoted by $\langle a_1, \ldots, a_n \rangle$. The diagonal forms $\langle 1, -a \rangle$, where $a \in k^*$, are denoted by $\langle\langle a \rangle\rangle$, and are said to be Pfister forms. The n-fold products of Pfister forms $\langle\langle a_i \rangle\rangle$ are denoted by $\langle\langle a_1, \ldots, a_n \rangle\rangle$. It is a classic theorem that, for any $p \geq 0$, the p-th power $I^p(k)$ of the fundamental ideal is generated by p-fold Pfister forms $\langle\langle a_1, \ldots, a_p \rangle\rangle$.

Define $s_1 : k^*/k^{*2} \to \overline{I}^1(k)$ by the assignment sending the class of a unit $a \in k^*$ to the class of the Pfister form $\langle\langle a \rangle\rangle$ in $\overline{I}^1(k)$. This is a well-defined isomorphism [EKM08, proof of Proposition 4.13], with inverse the signed determinant $\mathfrak{b} \mapsto (-1)^{\frac{\dim(\mathfrak{b})}{2}} \det(\mathfrak{b})$.

2.3. **Milnor K-theory.** Let k be a field. The n-th Milnor K-group $K_n^M(k)$ of k is defined to the abelian group defined by the following generators and relations: The generators are length n sequences $\{a_1, \ldots, a_n\}$ of units $a_i \in k^*$ (called symbols), and the relations are *multilinearity*

$$\{a_1, \ldots, a_{j-1}, xy, a_{j+1}, \ldots, a_n\} = \{a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_n\}$$
$$+ \{a_1, \ldots, a_{j-1}, y, a_{j+1}, \ldots, a_n\} \text{ for all } a_i, x, y \in k^* \text{ and } 1 \leq j \leq n;$$

and the *Steinberg relation* $\{a_1, \ldots, x, \ldots, 1 - x, \ldots, a_n\} = 0$ for all $a_i \in k^*$, and $x \in k - \{0, 1\}$.

2.4. **The Maps s^n and h^n.** Assume char $(k) \neq 2$. Consider the assignments

$$\{a_1, \ldots, a_n\} \mapsto \langle a_1, \ldots, a_n \rangle := s_1(a_1) \otimes \ldots \otimes s_1(a_n)$$

and

$$\{a_1, \ldots, a_n\} \mapsto (a_1, \ldots, a_n) := h_1(a_1) \cup \ldots \cup h_1(a_n)$$

that send the class of the symbol $\{a_1, \ldots, a_n\}$ to the class of the n-fold Pfister form $\langle\langle a_1, \ldots, a_n \rangle\rangle$ and the class of the symbol $\{a_1, \ldots, a_n\}$ to the Galois cohomology class $(a_1, \ldots, a_n)$, respectively. It is a classic fact that these maps respect the Steinberg and multilinearity relations, and send 2 to 0 [Mil70]. It follows from the

definition of the Milnor K-groups by generators and relations, that for all $n \geq 0$, the assignments above induce unique group homomorphisms

$$s^n : K_n^M(k)/2K_n^M(k) \to \overline{I}^n(k)$$

and

$$h^n : K_n^M(k)/2K_n^M(k) \to H_{Gal}^n(k, \mathbb{Z}/2).$$

We know that the homomorphism s^n is an isomorphism [OVV07], and from the work of Voevodsky [Voe03, Corollary 7.5], we know that h_n is an isomorphism.

Definition 2.1. Define $e_k^n : \overline{I}^n(k) \to H_{Gal}^n(k, \mathbb{Z}/2)$ to be the composition

$$\overline{I}^n(k) \xrightarrow{s_n^{-1}} K_n^M/2K_n^M(k) \xrightarrow{h^n} H^n(k, \mathbb{Z}/2\mathbb{Z}).$$

The homomorphism e_k^n is an isomorphism, and from the definition of s_n and h_n, the homomorphism e_k^n sends the class of a Pfister form $\langle\langle a_1, \ldots, a_n \rangle\rangle$ to the Galois cohomology class $(a_1, \ldots, a_n)$, hence agrees with e^n as defined by Arason in [Ara75, p.456].

2.5. **Cycle complexes with coefficients in $\overline{I}^n$.** We start by recalling what we mean by the residue and corestriction maps in the setting of Witt groups.

Definition 2.2. Let A be a DVR with fraction field K and residue field k, with char$(k) \neq 2$. For every uniformizing element $\pi \in A$, there is an associated group homomorphism

$$\partial_\pi : W(K) \to W(k),$$

satisfying

$$\partial_\pi(I^n(K)) \subset I^{n-1}(k),$$

and the induced homomorphism of abelian groups

$$\partial_\pi : \overline{I}^n(K) \to \overline{I}^{n-1}(k)$$

is independent of the choice of uniformizing element π [Ara75, Satz 3.1], hence is said to be *the residue homomorphism*.

Definition 2.3. For any finite field extension L/K and any non-trivial K-linear morphism $s : L \to K$ (see first sentence of the proof of Satz 3.3 for the fact that such a non-trivial K-linear morphism exists), the induced homomorphism on Witt groups $s_* : W(L) \to W(K)$ induces a homomorphism of groups

$$cor_{L/K} : \overline{I}^n(L) \to \overline{I}^n(K),$$

which is independent of s [Ara75, Satz 3.3], hence is defined to be *the corestriction for the finite field extension L/K*.

We proceed, as we did with the Kato complexes, by simply defining the yx-component of the differential.

Definition 2.4. Let X be an excellent scheme, finite dimensional of dimension d, with 2 invertible on X. Recall the notation of Definition 1.6. We define a sequence (one way to see that it is a complex is to use Arason's theorem below) of abelian groups

$$C^*(X, \overline{I}, n) := \bigoplus_{x \in X^0} \overline{I}^n(\kappa(x)) \xrightarrow{d} \bigoplus_{x \in X^1} \overline{I}^{n-1}(\kappa(x)) \xrightarrow{d} \cdots \cdots \xrightarrow{d} \bigoplus_{x \in X^d} \overline{I}^{n-d}(\kappa(x))$$

by defining the differentials componentwise. Set $\overline{I}^m(\kappa(x)) = 0$ for $m < 0$. For $y \in X^p$, $x \in X^{p+1}$, define the yx-component

$$d_{yx} : \overline{I}^i(\kappa(y)) \longrightarrow \overline{I}^{i-1}(\kappa(x))$$

as follows: If $x \notin \overline{\{y\}}$, then define $d_{yx} = 0$. If $x \in \overline{\{y\}}$, then define

$$d_{yx} := \sum_{x_i | x} cor_{\kappa(x_i)/\kappa(x)} \circ \partial^{x_i},$$

where ∂^{x_i} denotes the residue map of Definition 2.2, and $cor_{\kappa(x_i)/\kappa(x)}$ the core-striction of Definition 2.3. These complexes are called the *cycle complexes with coefficients in* $\overline{I}^n$.

Now we are able to state and prove Arason's Theorem.

Theorem 2.5 (Arason's Theorem). *Let X be a noetherian excellent scheme with 2 invertible in the global sections of X. If the Bloch-Kato conjecture is true, then the maps $e^n_{\kappa(x)}$ define, for all $n \geq 0$, an isomorphism of complexes $e^n : C(X, \overline{I}^n) \xrightarrow{\cong} C(X, H^n)$, from the cycle complex with coefficients in $\overline{I}^n$ to the Kato complex.*

Proof. Fix $n \geq 0$. The map $e^n : C(X, \overline{I}^n) \to C(X, H^n)$ is defined in the obvious way. On the degree i terms, it is

$$\bigoplus_{x \in X^i} \overline{I}^{n-i}(\kappa(x)) \overset{\oplus e^{n-i}_{\kappa(x)}}{\to} \bigoplus_{x \in X^i} H^{n-i}_{Gal}(\kappa(x), \mathbb{Z}/2\mathbb{Z})$$

where $\oplus e^{n-i}_{\kappa(x)}$ sums over the set X^i. To prove the theorem, we must prove that e^n defines a map of complexes.

Since the differentials are defined componentwise, it suffices to prove that the diagram below commutes

(2.4)
$$\begin{array}{ccc}
\overline{I}^i(\kappa(y)) & \xrightarrow{d_{yx}} & \overline{I}^{i-1}(\kappa(x)) \\
\downarrow{\scriptstyle e^i_{\kappa(y)}} & & \downarrow{\scriptstyle e^{i-1}_{\kappa(x)}} \\
H^i_{Gal}(\kappa(y), \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{d_{yx}} & H^{i-1}_{Gal}(\kappa(x), \mathbb{Z}/2\mathbb{Z})
\end{array}$$

for every pair of integers i, p, and every $y \in X^p$, $x \in X^{p+1}$. If $x \notin \overline{\{y\}}$, then both d_{yx} components are zero by definition, so the diagram commutes. If $x \in \overline{\{y\}}$, then, by definition,

$$d_{yx} := \sum_{x_i | x} cor_{\kappa(x_i)/\kappa(x)} \circ \partial^{x_i}$$

so

$$e^{i-1}_{\kappa(x)} \circ d_{yx} = \sum_{x_i | x} e^{i-1}_{\kappa(x)} \circ cor_{\kappa(x_i)/\kappa(x)} \circ \partial^{x_i}$$

because $e^i_{\kappa(x)}$ is a group homomorphism. Now we explain why, to prove that diagram 2.4 commutes, it suffices to show that both squares of the diagram below commute

(2.5)
$$\begin{array}{ccccc}
\overline{I}^i(\kappa(y)) & \xrightarrow{\partial^{x_i}} & \overline{I}^{i-1}(\kappa(x_i)) & \xrightarrow{cor_{\kappa(x_i)/\kappa(x)}} & \overline{I}^{i-1}(\kappa(x)) \\
\downarrow{\scriptstyle e^i_{\kappa(y)}} & & \downarrow{\scriptstyle e^{i-1}_{\kappa(x_i)}} & & \downarrow{\scriptstyle e^{i-1}_{\kappa(x)}} \\
H^i_{Gal}(\kappa(y), \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\partial^{x_i}} & H^{i-1}_{Gal}(\kappa(x_i), \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{cor_{\kappa(x_i)/\kappa(x)}} & H^{i-1}_{Gal}(\kappa(x), \mathbb{Z}/2\mathbb{Z})
\end{array}$$

for every x_i lying over x. Assume they do, that is,

$$e^{i-1}_{\kappa(x)} \circ cor_{\kappa(x_i)/\kappa(x)} \circ \partial^{x_i} = cor_{\kappa(x_i)/\kappa(x)} \circ \partial^{x_i} \circ e^i_{\kappa(y)}.$$

for every x_i lying over x. Hence,

$$
\begin{aligned}
d_{yx} \circ e^i_{\kappa(y)} &= \sum_{x_i | x} cor_{\kappa(x_i)/\kappa(x)} \circ \partial^{x_i} \circ e^i_{\kappa(y)} \\
&= \sum_{x_i | x} e^{i-1}_{\kappa(x)} \circ cor_{\kappa(x_i)/\kappa(x)} \circ \partial^{x_i} \\
&= e^{i-1}_{\kappa(x)} \circ d_{yx}.
\end{aligned}
$$

To finish the proof, recall the following results of Arason. For A a DVR with fraction field K and residue field k with $\operatorname{char}(k) \neq 2$, the diagram

$$
\begin{CD}
\overline{I}^n(K) @>\partial>> \overline{I}^{n-1}(k) \\
@Ve^n_K VV @VVe^{n-1}_k V \\
H^n(K, \mathbb{Z}/2\mathbb{Z}) @>\partial>> H^{n-1}(k, \mathbb{Z}/2\mathbb{Z})
\end{CD}
$$

is commutative [Ara75, Satz 4.11]. Additionally, when L/K is a finite field extension with $\operatorname{char}(K) \neq 2$, the diagram

$$
\begin{CD}
\overline{I}^n(L) @>cor_{L/K}>> \overline{I}^n(K) \\
@Ve^n_L VV @VVe^n_K V \\
H^n(L, \mathbb{Z}/2\mathbb{Z}) @>cor_{L/K}>> H^n(K, \mathbb{Z}/2\mathbb{Z})
\end{CD}
$$

is commutative [Ara75, Satz 4.18]. It follows that both squares of Diagram 2.5 commute, which concludes the proof. $\qquad\square$

3. Finiteness theorems for the shifted Witt groups

In this section, Arason's theorem is applied to Gille's graded Gersten-Witt spectral sequence. For more general schemes than for smooth varieties over fields, this allows the Witt groups to be related to the Zariski cohomology groups $H^p_{Zar}(X, \mathcal{H}^q)$, and hence, to the motivic cohomology groups.

3.1. Coniveau spectral sequence for coherent Witt groups.

Definition 3.1. For a noetherian scheme X with 2 invertible in its global sections, let $D^b_c(X)$ denote the derived category of bounded complexes of Zariski sheaves on X having coherent cohomology. A dualizing complex for X is defined to be a bounded complex K of injective coherent sheaves with the property that the natural morphism of complexes $\overline{\omega}^K$ (essentially the evaluation map, see [Gil07, §1.6] for a precise description) from an object M of $D^b_c(X)$ to its double dual $RHom(RHom(M, K), K)$ is an isomorphism (in $D^b_c(X)$). The *coherent Witt groups of* X are defined to be the Witt groups of the triangulated category with duality $(D^b_c(X), RHom(-, K), \overline{\omega}^K)$, and are denoted by $\widetilde{W}^n(X, K)$.

Remark 3.2. Let X be a noetherian scheme with 2 invertible in its global sections.

(a) When X is regular and separated, any injective resolution $I_\bullet$ of $\mathcal{O}_X$ yields a dualizing complex, and the quasi-isomorphism $\mathcal{O}_X \xrightarrow{\cong} I_\bullet$ induces an isomorphism $W^n(X) \xrightarrow{\cong} \widetilde{W}^n(X, I_\bullet)$ [Gil07, Example 2.4].

(b) Every dualizing complex $I_\bullet$ for X yields a function $\mu_I : X \to \mathbb{Z}$ [Gil07, Lemma 1.12 and following discussion], which can be used to obtain a filtration on the derived category. When X is regular and the dualizing complex

is given by an injective resolution of the structure sheaf, this function is exactly the usual codimension function $x \mapsto \mathrm{codim}(x)$ [Gil07, Example 1.13]. Hence, the filtration obtained is exactly the usual filtration by codimension of supports.

Let X be a noetherian regular separated $\mathbb{Z}[\frac{1}{2}]$-scheme of dimension d, and let $I_\bullet$ denote the dualizing complex obtained by taking an injective resolution of the structure sheaf $\mathcal{O}_X$. We briefly recall Gille's construction of *the coniveau spectral sequence* for coherent Witt groups [Gil07, §5.8]. Filtering by codimension of support defines a filtration $F^p D_c^b(X)$ on the bounded derived category $D_c^b(X)$. From this filtration, we obtain short exact sequences of triangulated categories with duality,

$$F^{p+1} D_c^b(X) \to F^p D_c^b(X) \to F^p D_c^b(X)/F^{p+1} D_c^b(X)$$

from which we obtain the long exact sequence of Witt groups below.

$$\to W^{p+q}(F^{p+1} D_c^b(X)) \to W^{p+q}(F^{p+1} D_c^b(X)) \to W^{p+q}(F^p D_c^b(X)/F^{p+1} D_c^b(X)) \to$$

From these exact sequences, an exact couple can be written down, which determines the spectral sequence below

$$(3.1) \qquad E_1^{p,q} := W^{p+q}(F^p D_c^b(X)/F^{p+1} D_c^b(X)) \Rightarrow W^{p+q}(X),$$

converging to the Witt groups of X (we have identified the coherent Witt groups with the usual Witt groups using Remark 3.2). The complexes $E_1^{*,q}$ appearing on the E_1-page, vanish for $q \neq 0 \mod 4$, and otherwise have the form [Gil07, §5.8]

$$(3.2) \quad C(X, W, \iota) := \bigoplus_{x \in X^0} W(\kappa(x)) \xrightarrow{d} \bigoplus_{x \in X^1} W(\kappa(x)) \xrightarrow{d} \cdots \cdots \xrightarrow{d} \bigoplus_{x \in X^d} W(\kappa(x))$$

where ι denotes the isomorphisms chosen to identify the complex $E_1^{*,q}$ with $C(X, W, \iota)$. The differentials may differ for different choices of isomorphisms ι.

One well known general fact about the shifted Witt groups of arithmetic schemes is the following easy corollary to the coniveau spectral sequence.

Corollary 3.3. *Let X be a noetherian regular $\mathbb{Z}[\frac{1}{2}]$-scheme of dimension d. If no residue field of X is formally real, then the Witt groups $W^n(X)$ are torsion groups.*

Proof. As no residue field of X is formally real, for each $x \in X$, the Witt group of the residue field $W(\kappa(x))$ is a torsion group [Sch85, Theorem 6.4 (ii)]. As arbitrary direct sums of torsion abelian groups are torsion, from the description in Equation 3.2 of the first page of the coniveau spectral sequence 3.1, we have that all the groups appearing on the first page are torsion groups. Since X is finite dimensional, the first page of the spectral sequence is bounded, hence convergent, so we have that the Witt groups are torsion. $\square$

The following proposition was proved by Stefan Gille [Gil07, §10], although (*b*) doesn't explicitly appear in [Gil07], so it requires proof.

Proposition 3.4 (Gille's Graded Gersten-Witt Spectral Sequence)**.** *Let X be a noetherian regular separated excellent $\mathbb{Z}[\frac{1}{2}]$-scheme.*

(a) *There is a spectral sequence (not necessarily convergent)*

$$E_1^{p,q} := H^{p+q}(C(X, I^p, \iota)/C(X, I^{p+1}, \iota)) \Rightarrow H^{p+q}(C(X, W, \iota))$$

where the abutment $H^{p+q}(C(X, W, \iota))$ is the cohomology of the Gersten-Witt complex, and the differential d_r has bidegree $(r, 1 - r)$.

(b) *The complexes $C(X, I^p, \iota)/C(X, I^{p+1}, \iota)$ are isomorphic to the cycle complexes $C(X, \overline{I}^p)$, hence $E_1^{p,q} = H^{p+q}(C(X, \overline{I}^p))$.*

Proof. First we recall briefly the construction of the spectral sequence . The differentials of the complex $C(X, W, \iota)$ respect the filtration by powers of the fundamental ideal [Gil07, Theorem 6.6], hence we obtain a filtered complex
(3.3)
$$C(X, I^n, \iota) := \bigoplus_{x \in X^0} I^n\left(\kappa\left(x\right)\right) \xrightarrow{d} \bigoplus_{x \in X^1} I^{n-1}\left(\kappa\left(x\right)\right) \xrightarrow{d} \cdots \cdots \xrightarrow{d} \bigoplus_{x \in X^n} I^0\left(\kappa\left(x\right)\right),$$

where we set $I^m\left(\kappa\left(x\right)\right) = W\left(\kappa\left(x\right)\right)$ when $m \leq 0$. The exact sequence of complexes

$$0 \to C(X, I^{n+1}, \iota) \to C(X, I^n, \iota) \to C(X, I^n, \iota)/C(X, I^{n+1}, \iota) \to 0$$

determines a long exact sequence in cohomology

$$\to H^{p+q}(C(X, I^{n+1}, \iota)) \to H^{p+q}(C(X, I^n, \iota)) \to H^{p+q}(C(X, I^p, \iota)/C(X, I^{p+1}, \iota)) \to$$

and setting $E_1^{p,q} := H^{p+q}(C(X, I^p, \iota)/C(X, I^{p+1}, \iota))$, we obtain an exact couple which gives the spectral sequence of the Proposition.

It remains to prove (b), that the quotient complexes obtained from $C(X, W, \iota)$ agree with the cycle complexes (note that the quotient complexes do not depend on the choices of isomorphisms ι [Gil07, Definition 7.4 and Lemma 7.5]). For a smooth variety over a field, the cycle complexes are exactly Rost's cycle complexes for the cycle module $\overline{I}^*$, so the assertion of (b) is exactly [Gil07, §10.7]. Nevertheless, in the general case the proof is identical. First, recall that for integral excellent rings the integral closure is finite in the fraction field (see Example 1.4). The components

$$d^{yx} : W\left(\kappa\left(y\right)\right) \longrightarrow W\left(\kappa\left(x\right)\right)$$

of the differentials of the complex $C(X, W, \iota)$ may be described as follows: If $x \notin \overline{\{y\}}$, then $d^{yx} = 0$. If $x \in \overline{\{y\}}$, then

$$d^{yx} := \sum_{x_i | x} cor_{\kappa(x_i)/\kappa(x)} \circ \partial^{x_i},$$

where ∂^{x_i} denotes the residue map of Definition 2.2 and $cor_{\kappa(x_i)/\kappa(x)}$ the corestriction of Definition 2.3 [Gil07, conjugate Lemma 7.2, Proposition 6.10 (taking $L = K$ and B to be the integral closure of A), and Proposition 6.5]. From this description of the differential, it follows that the n-th quotient complexes $C(X, I^n, \iota)/C(X, I^{n+1}, \iota)$ of the filtered complex $C(X, W, \iota)$ agree with the cycle complex $C(X, \overline{I}^n)$ of Definition 2.4. $\qquad\square$

Applying Arason's Theorem (Theorem 2.5), we obtain the following corollary.

Corollary 3.5. *Maintain the hypotheses of Proposition 3.4. The spectral sequence of Proposition 3.4 (not necessarily convergent) takes the form*

$$E_1^{p,q} := H^{p+q}(C(X, H^p)) \Rightarrow H^{p+q}(C(X, W, \iota)),$$

where $H^{p+q}(C(X, H^p))$ is the Kato cohomology of the p-th Kato complex.

Corollary 3.6. *Let X be a pure separated scheme that is smooth over $\mathbb{Z}[\frac{1}{2}]$, and suppose that no residue field of X is formally real. In this situation, the spectral sequence of Proposition 3.4 is convergent and takes the form*

$$E_1^{p,q} := H_{Zar}^{p+q}(X, \mathcal{H}^p) \Rightarrow H^{p+q}(C(X, W, \iota)),$$

where $\mathcal{H}^p$ denotes the Zariski sheaf associated to the presheaf $U \mapsto H_{ét}^p(U, \mathbb{Z}/2\mathbb{Z})$.

Proof. Applying Corollary 1.13, we have that $H^{p+q}(C(X, H^p)) = H_{Zar}^{p+q}(X, \mathcal{H}^p)$. Together with the previous corollary 3.5, this yields the description of the E_1-terms. To prove convergence, note that as no residue field of X is formally real, the cohomological dimension of X is $2\dim(X) + 1$ [SGA73, Exposé 5 §6 Theorem 6.2]. Hence, the groups $H_{Zar}^{p+q}(X, \mathcal{H}^p)$ vanish for $p > 2\dim(X) + 1$, from which it follows

that the first page of the spectral sequence is bounded, and therefore the spectral sequence strongly converges. □

Proposition 3.7. *Let X be a pure separated scheme that is smooth over a scheme S. Consider the following situations:*

    (a) $dim(X) \leq 1$, no residue field of X is formally real, and $S = \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$;

    (b) $dim(X) \leq 2$, X is quasi-projective, and $S = \mathrm{Spec}(\mathbb{F}_p)$ $(p > 2)$;

In either situation, the Witt groups $W^n(X)$ of X are finite.

Proof. In cases (a) and (b), the Kato cohomology is finite by Lemma 1.17 (a) and (b), respectively. Hence, applying the convergent spectral sequence of Corollary 3.6, we obtain finiteness of the Gersten-Witt complex $C(X, W, \iota)$. To finish the proof, use the convergent coniveau spectral sequence (Equation 3.1). □

Next we note the following consequence of Mayer-Vietoris.

Lemma 3.8. *Let $\mathcal{S}$ denote the category of noetherian regular separated $\mathbb{Z}[\frac{1}{2}]$-schemes.*

    (a) If, for any $X \in \mathcal{S}$, $W^n(X)$ is finite for all $n \in \mathbb{Z}$, then, for any $X \in \mathcal{S}$ and any line bundle L on X, $W^n(X, L)$ is finite for all $n \in \mathbb{Z}$.

    (b) If, for every connected $X \in \mathcal{S}$, $W^n(X)$ is finite for all $n \in \mathbb{Z}$, then, for every $X \in \mathcal{S}$, $W^n(X)$ is finite for all $n \in \mathbb{Z}$.

    (c) If, for every affine $X \in \mathcal{S}$, $W^n(X)$ is finite for all $n \in \mathbb{Z}$, then, for every $X \in \mathcal{S}$, $W^n(X)$ is finite for all $n \in \mathbb{Z}$.

Furthermore, the same statements are true with the Grothendieck-Witt groups $GW_n(X)$ in place of Witt groups.

Proof. For noetherian regular separated schemes with 2 invertible, Mayer-Vietoris holds for the Witt groups [Bal01b, Theorem 2.5], and for the Grothendieck-Witt groups [Sch10b, Theorem 1.1]. For (a), recall that as line bundles are locally free, an open cover of X on which L is trivial may be chosen. The lemma then follows by using Mayer-Vietoris and inducting on the number of open sets in the cover.

Next recall that the connected components of any locally noetherian X are open in X, and their intersection is empty. To prove (b), use Mayer-Vietoris, and proceed by induction on the number of connected components of X.

Recall that for any separated scheme, the intersection of any two affine subschemes is affine. To prove (c), use Mayer-Vietoris, and induct on the number of affine open sets necessary to cover X. □

The following well known Lemma will be used together with the previous one to reduce to X integral.

Lemma 3.9. *If X is a noetherian regular connected scheme, then X is integral.*

Proof. Let X be a noetherian regular connected scheme. As X is noetherian, it has only a finite number of irreducible components and every local ring $\mathcal{O}_{X,x}$ of X is also noetherian [Liu02, Chapter 2 Proposition 3.46(a)]. Since X has only a finite number of irreducible components, it is integral if and only if it is connected and integral at every point (i.e. $\mathcal{O}_{X,x}$ is an integral domain for every $x \in X$) [Liu02, Chapter 2 Exercise 4.4]. To finish the proof, recall that every regular noetherian local ring is a domain [Liu02, Chapter 4 Proposition 2.11]. □

Theorem 3.10 (Absolute Finiteness)**.** *Let X be a smooth variety over $\mathbb{F}_p$ $(p > 2)$, with $\dim(X) \leq 2$, and let L be a line bundle on X. In this situation, the Witt groups $W^n(X, L)$ are finite for all $n \in \mathbb{Z}$.*

Proof. We may assume that X is connected using Lemma 3.8 (b), hence, integral, using Lemma 3.9. Using 3.8 (c), we may assume X is affine, hence $X \to$ $\mathrm{Spec}(\mathbb{F}_p)$ is quasi-projective (any finite-type morphism between affine schemes is quasi-projective). Proposition 3.7 (b), and Lemma 3.8 (a) finish the proof. $\square$

Theorem 3.11. *Let X be a separated scheme that is smooth over $\mathbb{Z}[\frac{1}{2}]$, with no residue field of X formally real. Assume the Beilinson-Lichtenbaum conjecture (see Remark 1.19. Note that it is known for smooth varieties over fields). If the motivic cohomology groups $H_{mot}^m(X, \mathbb{Z}/2\mathbb{Z}(n))$ are finite for all $m, n \in \mathbb{Z}$, then the Witt groups $W^n(X)$ are finite for all $n \in \mathbb{Z}$.*

Proof. We may assume that X is connected using Lemma 3.8 (b), hence, integral using Lemma 3.9. Using that the Beilinson-Lichtenbaum conjecture holds and that the motivic cohomology groups $H_{mot}^m(X, \mathbb{Z}/2\mathbb{Z}(n))$ are all finite, we apply Lemma 1.20 to obtain that the Zariski cohomology groups $H_{Zar}^m(X, \mathcal{H}^n)$ are all finite. Since the spectral sequence of Corollary 3.6 is convergent, the cohomology groups of the Gersten-Witt complex $C(X, W, \iota)$ are finite. To finish the proof, use the coniveau spectral sequence converging to the Witt groups of X (Equation 3.1). $\square$

For ease of reference we include the following corollary.

Corollary 3.12. *Let X be a smooth variety over a finite field $\mathbb{F}_p$ $(p > 2)$, and let L be a line bundle on X. If the mod 2 motivic cohomology groups of X, $H_{mot}^m(X, \mathbb{Z}/2\mathbb{Z}(n))$, are finite for all $m, n \in \mathbb{Z}$, then the Witt groups $W^n(X, L)$ are finite for all $n \in \mathbb{Z}$.*

Finally, we note some partial converses to Theorem 3.11.

Theorem 3.13 (Relative Finiteness)**.** *Let X be a pure separated scheme that is smooth over a base scheme S. Consider the following situations (for (a), assume Beilinson-Lichtenbaum):*

- *(a) $dim(X) \leq 2$, no residue field of X is formally real, and $S = \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$;*
- *(b) $dim(X) \leq 3$, X is connected and proper over S, and $S = \mathrm{Spec}(\mathbb{F}_p)$ $(p > 2)$;*
- *(c) $dim(X) \leq 4$, X is connected and proper over S, and $S = \mathrm{Spec}(\mathbb{F}_p)$ $(p > 2)$.*

In situations (a) and (b):

- *(i)   The Witt groups $W^1(X)$ and $W^3(X)$ are finite;*
- *(ii) Finiteness of $W^0(X)$ is equivalent to finiteness of $W^2(X)$;*
- *(iii) $W^0(X)$ is finite if and only if the motivic cohomology groups $H_{mot}^p(X, \mathbb{Z}/2\mathbb{Z}(q))$ are finite for all $p, q \in \mathbb{Z}$.*

In situation (c):

- *(i)   The groups $W^0(X)$ and $CH^3(X)/2CH^3(X)$ are both finite if and only if the motivic cohomology groups $H_{mot}^p(X, \mathbb{Z}/2\mathbb{Z}(q))$ are finite for all $p, q \in \mathbb{Z}$.*

Proof. In all situations, finiteness of the Witt groups follows from finiteness of motivic cohomology by Theorem 3.11, so we will only prove the other direction.

Assume that we are in situation (a). First we prove (i). To prove that $W^1(X)$ is finite, we will prove that the group $H^1(C(X, W, \iota))$ is finite, and then use the coniveau spectral sequence (Equation 3.1). To prove that $H^1(C(X, W, \iota))$ is finite, considering the spectral sequence of Corollary 3.6, it suffices to prove that the groups on the on the $p + q = 1$ diagonal of the first page of the spectral sequence, $H_{Zar}^1(X, \mathcal{H}^p)$ for $p \geq 0$, are finite. This was shown in Lemma 1.21 (a) (i). The proof of finiteness of $W^3(X)$ is identical.

Now we prove (ii). Assume that $W^0(X)$ is finite. Considering the shape of the coniveau spectral sequence (Equation 3.1), this implies $H^0(C(X, W, \iota))$ is finite. Considering the spectral sequence of Corollary 3.6, if we prove that all the groups

$H^2_{Zar}(X, \mathcal{H}^p)$ are finite, this will prove that $H^2(C(X, W, \iota))$ is finite, hence prove that $W^2(X)$ is finite. To accomplish this, using Lemma 1.21 (a) (i), it suffices to prove that the groups $H^0_{Zar}(X, \mathcal{H}^3)$, $H^0_{Zar}(X, \mathcal{H}^4)$, and $H^0_{Zar}(X, \mathcal{H}^5)$ are finite. Note that once we prove this, it will also finish the proof of (iii). Consider the spectral sequence of corollary 3.6. Since $H^0_{Zar}(X, \mathcal{H}^5)$ has no non-zero differentials entering or leaving it, it is stable, hence finite by finiteness of $W^0(X)$. There is one possibly non-zero differential leaving the group $H^0_{Zar}(X, \mathcal{H}^4)$. It is the differential $d^{4,-4}_1 : E^{4,-4}_1 = H^0_{Zar}(X, \mathcal{H}^4) \to E^{5,-4}_1 = H^1_{Zar}(X, \mathcal{H}^5)$. Since the kernel of $d^{0,4}_1$ is stable, and is on the 0-th diagonal, it is finite by finiteness of $W^0(X)$. So finiteness of $H^1_{Zar}(X, \mathcal{H}^4)$ follows from finiteness of $H^1_{Zar}(X, \mathcal{H}^5)$ (Lemma 1.21 (a) (i)). Next, we will prove that $H^0_{Zar}(X, \mathcal{H}^3)$ is finite. First, consider the differential $d^{3,-3}_1 : E^{3,-3}_1 = H^0_{Zar}(X, \mathcal{H}^3) \to E^{4,-3}_1 = H^1_{Zar}(X, \mathcal{H}^4)$. Since $H^1_{Zar}(X, \mathcal{H}^4)$ is finite, $H^0_{Zar}(X, \mathcal{H}^3)$ is finite if and only if the kernel of $d^{3,-3}_1$ is finite. The kernel of $d^{3,-3}_1$ equals $E^{3,-3}_2$. Consider the differential $d^{3,-3}_2 : E^{3,-3}_2 \to E^{5,-4}_2$. Since $H^0_{Zar}(X, \mathcal{H}^5)$ is finite, its quotient $E^{5,-4}_2$ is also finite. As the kernel of $d^{3,-3}_2$ is on the 0-th diagonal of the stable page, and $W^0(X)$ is finite, we obtain finiteness of $E^{3,-3}_2$. Thus, proving that $W^2(X)$ is finite. The proof that finiteness of $W^2(X)$ implies finiteness of $W^0(X)$ is identical.

Assume that we are in situation (b). First we prove prove (i), finiteness of $W^1(X)$. As in situation (a), it suffices to prove that the groups $H^1_{Zar}(X, \mathcal{H}^p)$ are finite, for $p \geq 0$. This was shown in Lemma 1.21 (b) (i). Similarly, we have that $W^3(X)$ is finite. Next, to prove (ii), assume $W^0(X)$ is finite. Consider the spectral sequence of Corollary 3.6. As $E^{3,-3}_1 = H^0_{Zar}(X, \mathcal{H}^3)$ is on the 0-th diagonal of the stable page, it is finite. So using Lemma 1.21 (b) (i) and (ii), this proves (iii), and we have that all the groups $H^2_{Zar}(X, \mathcal{H}^p)$, for $p \geq 0$, are finite, which proves finiteness of $W^2(X)$. The other direction is identical.

Finally, assume we are in situation (c). Consider the spectral sequence of Corollary 3.6. By hypothesis $W^0(X)$ is finite, so the stable term $E^{4,-4}_1 = H^0_{Zar}(X, \mathcal{H}^4)$ is finite. Additionally, by hypothesis $CH^3(X)/2CH^3(X) = H^3_{Zar}(X, \mathcal{H}^3)$ is finite, hence $H^1_{Zar}(X, \mathcal{H}^4)$ is finite (Lemma 1.21 (c) (i)). Now consider the differential $d^{3,-3}_1 : H^0_{Zar}(X, \mathcal{H}^3) \to H^1_{Zar}(X, \mathcal{H}^4)$. As the kernel of $d^{3,-3}_1$ is on the 0-th diagonal of the stable page, it is finite by finiteness of $W^1(X)$. Therefore, $H^0_{Zar}(X, \mathcal{H}^3)$ is finite, which is enough to finish the proof using Lemma 1.21 (c) (i). $\square$

4. Finite generation theorems for the higher Grothendieck-Witt groups

In this section, we prove finite generation theorems for the Grothendieck-Witt groups of arithmetic schemes. Let X be a noetherian separated scheme over $\mathbb{Z}[\frac{1}{2}]$, and let L be a line bundle on X. Let $Ch^bVect(X)$ denote the category of bounded chain complexes of vector bundles on X. By shifting L, for each $n \in \mathbb{Z}$, we obtain a duality $Hom(-, L[n])$ on $Ch^bVect(X)$. We work with Schlichting's Grothendieck-Witt spectrum $GW^n(X, L)$ [Scha] associated to the category $Ch^bVect(X)$, equipped with the duality $Hom(-, L[n])$ and with quasi-isomorphisms as weak equivalences. Its m-th homotopy groups are denoted by $GW^n_m(X, L)$, and are said to be the *Grothendieck-Witt groups of X with coefficients in L*. These groups are 4-periodic in n, $GW^n_m(X, L) \simeq GW^{n+4}_m(X, L)$. The negative Grothendieck-Witt groups, that is, the negative homotopy groups of the Grothendieck-Witt spectrum, agree with the Witt groups $GW^0_{-m}(X) \simeq W^m(X)$, for $m > 0$. For these facts, see [Scha].

Proposition 4.1. *Let X be a noetherian regular separated $\mathbb{Z}[\frac{1}{2}]$-scheme. For every $n \in \mathbb{Z}$, there is a long exact sequence of abelian groups*

$$\cdots \to GW_m^n(X) \to GW_{m-1}^{n-1}(X) \xrightarrow{F} K_{m-1}(X) \xrightarrow{H} GW_{m-1}^n(X) \to \cdots$$

which may be completed to end in

$$\cdots GW_0^{n-1}(X) \to K_0(X) \xrightarrow{H} GW_0^n(X) \to W^n(X) \to 0$$

Proof. It is proved in [Scha] that the sequence of spectra

$$GW^{n-1}(X) \xrightarrow{F} K(X) \xrightarrow{H} GW^n(X)$$

is a homotopy fibration, where $K(X)$ is the algebraic K-theory spectrum whose homotopy groups are the higher algebraic K-groups. Hence, it determines the long exact sequence of the proposition, with one exception. To complete the long exact sequence to the form stated in the proposition, we use the isomorphism $GW_{-m}^0(X) \simeq W^m(X)$ ($m > 0$) [Scha], which gives $GW_{-1}^0(X) \simeq W^1(X)$. Shifting the duality on both sides $(n-1)$-times, we obtain $GW_{-1}^{n-1}(X) \simeq W^n(X)$. Since there are no negative K-groups of X, the map $GW_0^n(X) \to W^n(X)$ is surjective as asserted. $\qquad\square$

Karoubi induction is a well known means of proving the corollary below. We give the corollary the name "Schlichting" induction because the argument is different than the usual Karoubi induction argument (i.e. it uses the fibration above), and it was suggested to the author by Schlichting.

Corollary 4.2 ("Schlichting" Induction). *Maintain the hypothesis of the previous Proposition. Assume that the groups $K_m(X)$ are finitely generated for all $m \in \mathbb{Z}$. If the Witt groups $W^n(X)$ are finitely generated for all $n \in \mathbb{Z}$, then the Grothendieck-Witt groups $GW_m^n(X)$ are finitely generated for all $m, n \in \mathbb{Z}$.*

Proof. We will prove the result by induction on m. The base case is $m = -1$, finite generation of the Witt groups $W^n(X) \simeq GW_{-1}^{n-1}$, for all $n \in \mathbb{Z}$. For the induction step, suppose that $GW_{m-1}^{n-1}(X)$ is finitely generated, for all $n \in \mathbb{Z}$. Using the fibration sequence of Proposition 4.1, and finite generation of the algebraic K-theory groups $K_m(X)$, we obtain that $GW_m^n(X)$ is also finitely generated, for all $n \in \mathbb{Z}$. $\qquad\square$

Theorem 4.3. *Let X be a separated scheme that is smooth over $\mathbb{Z}[\frac{1}{2}]$, with no residue field of X formally real (e.g. , a smooth variety over a finite field $\mathbb{F}_p$ ($p > 2$)), and let L be a line bundle on X. If $\dim(X) \leq 1$, then the Grothendieck-Witt groups $GW_m^n(X, L)$ are finitely generated groups.*

Proof. We may assume that X is connected using Lemma 3.8 (b), hence, integral using using Lemma 3.9. Under the hypotheses on X, the algebraic K-groups $K_m(X)$ are finitely generated for all $m \in \mathbb{Z}$ [Kah05, §4.7.1 Proposition 38 (b)]. So, the result follows from Corollary 4.2 and Theorem 3.7 (a) (use Lemma 3.8 to get the result for any line bundle L on X). $\qquad\square$

We have the following conditional result.

Theorem 4.4. *Let X be a separated scheme that is smooth over $\mathbb{Z}[\frac{1}{2}]$, with no residue field of X formally real (e.g. , a smooth variety over a finite field $\mathbb{F}_p$ ($p > 2$)), and let L be a line bundle on X. Assume the Beilinson-Lichtenbaum conjecture holds (see Remark 1.19, note this is known for smooth varieties over fields). If the motivic cohomology groups $H_{mot}^m(X, \mathbb{Z}(n))$ are finitely generated for all $m, n \in \mathbb{Z}$, then the Grothendieck-Witt groups $GW_m^n(X, L)$ are finitely generated for all $m, n \in \mathbb{Z}$.*

Proof. We may assume that X is connected using Lemma 3.8 (b), hence, integral using Lemma 3.9. After applying the Atiyah-Hirzebruch spectral sequence converging to K-theory [Kah05, 4.3.2 Equation (4.6) and the final paragraph of §(4.6)], we obtain that K-theory $K_m(X)$ is finitely generated for all $m \in \mathbb{Z}$. Multiplication by 2 defines a short exact sequence of motivic sheaves

$$0 \to \mathbb{Z}(n) \xrightarrow{2} \mathbb{Z}(n) \to \mathbb{Z}/2\mathbb{Z}(n) \to 0,$$

for every $n \in \mathbb{Z}$. This induces a long exact sequence

$$\cdots \to H^m_{mot}(X, \mathbb{Z}(n)) \to H^m_{mot}(X, \mathbb{Z}(n)) \to H^m_{mot}(X, \mathbb{Z}/2\mathbb{Z}(n)) \to \cdots$$

of motivic cohomology groups. Using the hypothesis that the motivic cohomology groups $H^m_{mot}(X, \mathbb{Z}(n))$ are finitely generated, it follows that the groups $H^m_{mot}(X, \mathbb{Z}/2\mathbb{Z}(n))$ are also finitely generated, hence finite, as they are torsion. By Theorem 3.11(a), the Witt groups $W^n(X)$ are finite. Therefore, Corollary 4.2 finishes the proof (use Lemma 3.8 to get the result for any line bundle on X). $\qquad\square$

5. Finiteness of the d-th Chow-Witt group

Throughout this section, X will denote a variety (i.e. separated and of finite type) that is smooth over a field k (char$(k) \neq 2$). First, we recall the definition of the Chow-Witt groups (a.k.a. Chow groups of oriented cycles). The n-th cycle complex with coefficients in Milnor K-theory [Kat86b] is a complex consisting of Milnor K-groups

$$C(X, K^M_n) := \bigoplus_{x \in X^0} K^M_n(\kappa(x)) \xrightarrow{d} \bigoplus_{x \in X^1} K^M_{n-1}(\kappa(x)) \xrightarrow{d} \cdots \cdots \xrightarrow{d} \bigoplus_{x \in X^d} K^M_{n-d}(\kappa(x))$$

with differential defined componentwise, exactly as was done in Definition 1.7, however using the the residue morphism for Milnor K-theory. The natural map $s^n : K^M_n(k) \to \overline{I}^n(k)$ (see Section 2.4), defined for every field k with char$(k) \neq 2$, induces a map of complexes $s^n : C(X, K^M_n) \to C(X, \overline{I}^n)$ (e.g., see Theorem 10.2.6 in Fasel's Thesis, or [Fas08]), where $C(X, \overline{I}^n)$ is the complex of Definition 2.4. To obtain the complex

$$C(X, I^n, \omega_{X/k}) := \bigoplus_{x \in X^0} I^n(\kappa(x); \Lambda^0) \to \cdots \to \bigoplus_{x \in X^d} I^{n-d}(\kappa(x); \Lambda^d),$$

that is also needed to define the Chow-Witt groups, where $\Lambda^i := \Lambda^i((m_x/m_x^2)^*)$, one begins with Schmid's Gersten-Witt complex [Schb, Satz 3.3.2]

$$C(X, W, \omega_{X/k}) := \bigoplus_{x \in X^0} W(\kappa(x); \Lambda^0) \to \cdots \to \bigoplus_{x \in X^d} W(\kappa(x); \Lambda^d),$$

and filters it by the powers of the fundamental ideal, for example, see [BM00]. Recall that for any field k, and any one-dimensional k-vector space L, a choice of generator for L defines an isomorphism $W(k) \to W(k, L)$, and by definition $I^n(k; L) := I^n(k) \cdot W(k; L)$, as $I^n(k; L)$ does not depend on the choice of isomorphism (e.g., see Lemma A.1.2 in Fasel's Thesis, or [Fas08]). The quotient complexes $C(X, I^n, \omega_{X/k})/C(X, I^{n+1}, \omega_{X/k})$ will be denoted simply by $C(X, \overline{I}^n)$, as they are in fact isomorphic (e.g., see Lemma A.1.3 in Fasel's Thesis, or [Fas08]).

Definition 5.1. Define the complex $C(X, J^n)$ to be the fiber product of the complexes $C(X, I^n, \omega_{X/k})$ and $C(X, K^M_n)$ over $C(X, \overline{I}^n)$. Hence, $C(X, J^n)$ lives in a

diagram

$$\begin{array}{ccc} C(X, J^n) & \longrightarrow & C(X, I^n, \omega_{X/k}) \\ \downarrow & & \downarrow \\ C(X, K_n^M) & \xrightarrow{\ s^n\ } & C(X, \overline{I}^n) \end{array}$$

where the map from $C(X, I^n, \omega_{X/k})$ to $C(X, \overline{I}^n)$ is the quotient map. For any $n \geq 0$, the n-th Chow-Witt group $\widetilde{CH}^n(X)$ is defined to be the n-th cohomology group of the complex $C(X, J^n)$.

The following lemma is a slight variation on an argument of Gille.

Lemma 5.2. [Gil07, Proof of Proposition 10.3] *Suppose that the base field k is a finite field $\mathbb{F}_p$ ($p > 2$), and $\dim(X) = d$. Then, for all $j \geq 0$, the complex $C(X, I^{j+d+2}, \omega_{X/k})$ vanishes, and the quotient map $C(X, I^{d+1}, \omega_{X/k}) \xrightarrow{\simeq} C(X, \overline{I}^{d+1})$ is an isomorphism of complexes.*

Proof. Let $x \in X^p$ be a codimension p point of X. By Lemma 1.14, the cohomological 2-dimension $\mathrm{cd}_2(\kappa(x))$ of the residue field $\kappa(x)$ of x satisfies $\mathrm{cd}_2(\kappa(x)) \leq 1+d-p$. Since the map $e_k^i : \overline{I}^i(k) \to H_{Gal}^i(k, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism (see Definition 2.1) for every field k, $\overline{I}^{2+d-p}(\kappa(x)) = 0$. It follows that $I^{2+d-p}(\kappa(x)) = \cap_{n \geq 2+d-p} I^n(\kappa(x))$. By the Arason-Pfister Haupsatz, $0 = \cap_{n \geq 0} I^n(\kappa(x))$. Therefore, $I^{2+d-p}(\kappa(x)) = 0$, hence, by definition,

$$I^{2+d-p}(\kappa(x); \Lambda^p((m_x/m_x^2)^*)) := I^{2+d-p}(\kappa(x)) \cdot W(\kappa(x); \Lambda^p((m_x/m_x^2)^*)) = 0$$

and from this, for all $j \geq 0$, $C(X, \overline{I}^{j+d+2}, \omega_{X/k}) = 0$ follows. Then, the exact sequence of complexes

$$0 \to C(X, I^{d+2}, \omega_{X/k}) \to C(X, I^{d+1}, \omega_{X/k}) \to C(X, \overline{I}^{d+1}) \to 0$$

degenerates into the desired isomorphism, finishing the proof of the lemma.   □

Now we are ready to state and prove the finiteness theorem.

Theorem 5.3. *Let X be a smooth and quasi-projective variety over a finite field $\mathbb{F}_p$ ($p > 2$), pure dimensional of dimension d. Then the d-th Chow-Witt group $\widetilde{CH}^d(X)$ is finite.*

Proof. Recall, it follows from the definition that there is always an exact sequence

$$CH^d(X) \to \widetilde{CH}^d(X) \to H^d(X, I^d) \to 0,$$

and for any quasi-projective variety over a finite field, the group $CH^d(X)$ is finite [KS10, Corollary 9.4 (1)]. So, the proof reduces to proving that $H^d(C(X, I^d, \omega_{X/k}))$ is finite. From the short exact sequence of complexes

$$0 \to C(X, I^{d+1}, \omega_{X/k}) \to C(X, I^d, \omega_{X/k}) \to C(X, \overline{I}^d) \to 0,$$

we obtain the long exact sequence in cohomology

$$\cdots \to H^d(C(X, I^{d+1}, \omega_{X/k})) \to H^d(C(X, I^d, \omega_{X/k})) \to H^d(C(X, \overline{I}^d)) \to \cdots$$

From Arason's theorem (Theorem 2.5), it follows that $H^d(C(X, \overline{I}^d))$ and $H^d(C(X, \overline{I}^{d+1}))$ are isomorphic to the Kato cohomology groups $H^d(C(X, H^d))$ and $H^d(X, H^{d+1})$, respectively. The latter are finite by Lemma 1.17 (d) and (c), respectively. We conclude the proof by identifying $H^d(C(X, I^{d+1}, \omega_{X/k}))$ with $H^d(C(X, \overline{I}^{d+1}))$ using Lemma 5.2.   □

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Department of Mathematics, Louisiana State University, Baton Rouge, LA
E-mail address: `jjaco19@math.lsu.edu`